3SAT: → NP

INPUT: φ in 3CNF

DECIDE: If φ is satisfiable → Cook-Levin: NP-complete

$\forall A \in NP: A \leq_p 3SAT$

2SAT: → in P

INPUT: φ in 2CNF

DECIDE: –||–

$\bigwedge_{i \in n} (x_{i_1} \to x_{i_2})$

Claim: φ is sat $\iff$ G_φ has no vertex x s.t. there's a path from x to $\neg x$ and vice versa

XOR-SAT

INPUT: φ, a conjunction of XOR clauses

DECIDE: –||–

$x + (1+y) + z = 1$.

$(x \oplus \neg y \oplus z)$

Deciding φ is the same as solving sim. equations over $\mathbb{F}_2$ → can be done in P with Gaussian elimination (contains $=$)

Def:
Let λ be a finite relational language, and fix a finite λ-structure Γ.

CSP(Γ):

Input: a finite λ-structure A

Decide: $A \to \Gamma$, where $A \to \Gamma :\iff \exists f: A \to \Gamma$ s.t. for all $R \in \lambda$ and all $\bar{a} \in R^A \Rightarrow f(\bar{a}) \in R^\Gamma$.

Eg: • $3SAT = CSP(\{0,1\}, R, \neq)$ $\in$ NP-complete
where $R = \{0,1\}^3 \setminus \{0,0,0\}$

• $2SAT = CSP(\{0,1\}, \leq, \neq) \in P$
solved by local propagation of information

• $XORSAT = CSP(\{0,1\}, \{(0,0,0),(1,1,0),(0,1,1),(1,0,1)\}, \neq)$
$\in P$ solved by Gaussian elimination

<u>Schaefer's Theorem (1978)</u>

Let Γ be an λ-structure over $\{0,1\}$. If $CSP(\Gamma)$ is not NP-complete, then of the following occurs

1. all relations contain the 0-vector
2. -11- 1-vector eg $R = \{0,1\}^n \setminus \{(1,1,0,\dots)\}$
3. -11- are intersections of Horn clauses
4. -11- dual Horn clauses
5. -11- can be written as conjunctions of binary clauses
6. all relations are solutions to simultaneous linear equations over $\mathbb{F}_2$.

and $CSP(\Gamma)$ is in P.

<u>Hell-Nešetřil (1990):</u>

For an undirected graph G, $CSP(G)$ is in P if G is bipartite, and o/w $CSP(G)$ is NP-complete.

$$\left.\begin{array}{l} G \to K_2 := \bullet\!-\!\bullet \\ K_2 \to G \end{array}\right\} CSP(G) = CSP(K_2)$$

Conjecture (Feder-Vardi, 1998)
For finite Γ, CSP(Γ) is either in P, or it is NP-complete.

Ladner's Theorem (1975)
If $P \neq NP$, then there are problems in $NP \setminus P$ which are not NP-complete.

Th: For every σ-structure Γ, there is some digraph G s.t. CSP(Γ) is poly-time equiv. to CSP(G).

Def: (Generalised group problem)
Fix G a finite group. Define a relational structure $\hat{G}$ on G by adding a relation gH for every $H \leq G^n$ and $g \in G$.
Th: CSP($\hat{G}$) is in P.

Tractable cases:
1. Bounded width; solved local propagation of information
2. Subgroup problems; solved by Gaussian elimination type algorithms

Observation: The complexity of CSP(Γ) depends on the structure of the operations $\Gamma^k \to \Gamma$ that preserve the relations of Γ.

Algebraic approach

Def: Let Γ be a set. We say that a k-ary operation $f: \Gamma^k \to \Gamma$ preserves an n-ary relation $R \subseteq \Gamma^n$ if R is a subalgebra of $(\Gamma, f)^n$, i.e.

if: $\begin{pmatrix} x_{11} \\ \vdots \\ x_{1n} \end{pmatrix}, \dots, \begin{pmatrix} x_{k1} \\ \vdots \\ x_{kn} \end{pmatrix} \in R$, then:

$$f\left(\begin{pmatrix} x_{11} \\ \vdots \\ x_{1n} \end{pmatrix}, \dots, \begin{pmatrix} x_{k1} \\ \vdots \\ x_{kn} \end{pmatrix} \right) = \begin{pmatrix} f(x_{11}, \dots, x_{k1}) \\ \vdots \\ f(x_{1n}, \dots, x_{kn}) \end{pmatrix} \in R.$$

An operation f is a polymorphism of an Δ-structure Γ if f preserves all relations of Γ.

Eg: The operation $(x,y,z) \mapsto xy^{-1}z$ is a polymorphism of $\mathcal{G}$.

Observation: The algebra $Pol(\Gamma)$ of polymorphisms of Γ is a clone, ie it contains the projections $\pi_i^k: \Gamma^k \to \Gamma$ and it's closed under composition, ie f is k-ary and $g_1, \dots, g_k$ are ℓ-ary then:

$$f \circ (g_1, \dots, g_k): (x_1, \dots, x_\ell) \mapsto f(g_1(\bar{x}), \dots, g_k(\bar{x})) \in Pol(\Gamma).$$

Def: For an algebra $\mathbb{A}$, write $Inv(\mathbb{A})$ for the set of relations $R \subseteq A^n$ which are preserved by the operations in $\mathbb{A}$.

Th: (Geiger, Bodnarčuk) (Γ is ω-categorical)

$$Inv(Pol(\Gamma)) = \langle \Gamma \rangle_{pp} \qquad \boxed{\langle \Gamma \rangle_{FO} = Inv(Aut(\Gamma))}$$

where $\langle \Gamma \rangle_{pp}$ is the expansion of Γ by all pp-definable subsets.

Th (Jeavons):
Let Γ, Γ' be two structures on the same domain.
If $Pol(\Gamma) \leq Pol(\Gamma')$ then Γ' is a pp-reduct of Γ (ie $\Gamma' \leq \langle \Gamma \rangle_{pp}$) and

$$CSP(\Gamma') \leq_p CSP(\Gamma).$$

$\therefore$ The clone of polymorphisms of Γ determine the complexity of $CSP(\Gamma)$.

$\boxed{\text{In fact: The equational identities of } Pol(\Gamma) \text{ determine the complexity of } \Gamma}$

- Bulatov - Dalman (2006): A Gaussian elimination type algorithm solves in poly time any CSP with a Maltsev polymorphism, ie an operation satisfying

$$p(x,y,y) = p(y,y,x) = x.$$

- Theorem (Bulatov - Zhuk) 2017:
  - If $Pol(\Gamma)$ has a cyclic term, ie a function of arity p (where p is prime) s.t.
    $$f(x_1, \ldots, x_p) = f(x_2, \ldots, x_p, x_1)$$
    then $CSP(\Gamma)$ is in P
  - o/w $CSP(\Gamma)$ is NP-complete.