3.3: Bi-Interpretations

Def: Two structures $\mathcal{C}$ and $\mathcal{D}$ with a primitive positive interpretation I of $\mathcal{C}$ in $\mathcal{D}$ and a p.p. interpretation J of $\mathcal{D}$ in $\mathcal{C}$ are called mutually p.p. interpretable. If both $I \circ J$ and $J \circ I$ are pp-homotopic to the identity interpretation (of $\mathcal{D}$ and $\mathcal{C}$ respectively) then we say that $\mathcal{C}$ and $\mathcal{D}$ are p.p bi-interpretable

where:

$I \circ J$ is pp-homotopic to the identity interpretation $\iff$ the relation $\{(x, \bar{y}) : I \circ J(\bar{y}) = x\}$ is p.p definable in $\mathcal{D}$.

Example 3.3.3: Let $\mathbb{I}$ be the set of all non-empty, closed, bounded intervals over $\mathbb{Q}$, and $m \subseteq \mathbb{I}^2$ the relation

$$\{([x_1, x_2], [y_1, y_2]) \in \mathbb{I}^2 : \quad x_2 = y_1\}.$$

Define a 2-dimensional interpretation I of $\mathbb{I}$ in $\mathbb{Q}$:

- $\mathrm{dom}(I) = \{(x,y) \in \mathbb{Q}^2 : \quad x < y\}$, so $T_I(x,y) := x < y$
- $I : \mathrm{dom}(I) \to \mathbb{I}$ maps (x,y) to $[x,y]$
- $=_I (x_1, x_2, y_1, y_2) := (x_1 = y_1) \wedge (x_2 = y_2)$
- $m_I (x_1, x_2, y_1, y_2) := (x_2 = y_1)$

Clearly this is p.p.

Likewise, define a 1-dim. interpretation J of $\mathbb{Q}$ in $\mathbb{I}$:

- $\mathrm{dom}(J) = \mathbb{I}$, so $T_J := T$
- $J : [x,y] \mapsto x$
- $=_J (a,b) := \exists c\, (m(c,a) \wedge m(c,b))$
- $<_J (a,b) := \exists c, d\, (m(c,a) \wedge m(c,d) \wedge m(d,b))$

This is also p.p. We also show that $J \circ I$ and $I \circ J$ are pp-homotopic to the identities.

$J \circ I \sim \mathrm{id}_{\mathbb{Q}}$: $\{(x_1, x_2, y) : \underbrace{\underbrace{J \circ I(x_1, x_2)}_{J([x_1, x_2])}}_{x_1} = y\} \subseteq \mathbb{Q}^2$ is p.p. definable by $x_1 = y$.

$I \circ J \sim \mathrm{id}_{\mathbb{I}}$: $\{(a, b, c) : \underbrace{I(J(a), J(b))}_{[a_1, b_1]} = c\} \subseteq \mathbb{I}^2$ is p.p definable by $a =_J c \wedge m(c,b)$ //

Example 3.3.4: The structures $\mathcal{C} := (\mathbb{N}^2, \{(x,y),(u,v) : x = u\})$ and $\mathcal{D} := (\mathbb{N}, =)$ are mutually p.p. interpretable but not even first-order bi-interpretable.

Looking ahead: Let $\mathcal{C}, \mathcal{D}$ be two ω-categorical structures. Then:

$\mathcal{C}, \mathcal{D}$ are bi-interpretable $\iff$ $\mathrm{Aut}(\mathcal{C})$, $\mathrm{Aut}(\mathcal{D})$ are isomorphic as topological groups.

$\mathcal{C}, \mathcal{D}$ are pp bi-interpretable $\Leftrightarrow$ $\mathrm{Pol}(\mathcal{C})$, $\mathrm{Pol}(\mathcal{D})$ are isomorphic as topological clones.

Def: A structure $\mathcal{B}$ has essentially infinite signature if every $\mathcal{C}$ that is p.p. interdefinable with $\mathcal{B}$ has infinite signature.

where:
$\mathcal{B}, \mathcal{C}$ are interdefinable $\Leftrightarrow$ they are on the same domain, all relations of $\mathcal{B}$ are definable in $\mathcal{C}$ and vice versa.

Proposition 3.3.6: Let $\mathcal{B}, \mathcal{C}$ be p.p. bi-interpretable. Then $\mathcal{B}$ has essentially infinite signature $\Leftrightarrow$ $\mathcal{C}$ does.

Proof: It suffices to show that if $\mathcal{C}$ has finite signature, then $\mathcal{B}$ is pp interdefinable with some $\mathcal{B}'$ in a finite signature. We'll choose $\mathcal{B}'$ to be some finite reduct of $\mathcal{B}$.

Let τ be the signature of $\mathcal{B}$, d_1 the dimension of the interpretation I_1 of $\mathcal{C}$ in $\mathcal{B}$, and d_2 -"- of I_2 of $\mathcal{B}$ in $\mathcal{C}$. Let θ be the pp formula witnessing that $I_2 \circ I_1$ is pp homotopic to $\mathrm{id}_{\mathcal{B}}$, ie

$$\mathcal{B} \models \theta(a, b_{11}, \ldots, b_{d_1 d_2}) \Leftrightarrow a = I_2(I_1(b_{11}, \ldots, b_{d_1 1}), \ldots, I_1(b_{1 d_2}, \ldots, b_{d_1 d_2}))$$

Let $\sigma \subseteq \tau$ be the set of relation symbols that appear in θ and in all the formulas of the interpretation of $\mathcal{C}$ in $\mathcal{B}$. We argue that $\mathcal{B}$ has a pp definition in its σ-reduct $\mathcal{B}'$.

Let φ be an atomic τ-formula with free variables $x_1, \ldots, x_k$. Consider

$$\phi(x_1, \ldots, x_k) := \exists y^1_{11}, \ldots, y^k_{d_1 d_2} \left[\bigwedge_{i \leq k} \theta(x_i, y^i_{11}, \ldots, y^i_{d_1 d_2}) \right.$$
$$\left. \wedge\ \varphi_{I_2 \circ I_1}(y^1_{11}, \ldots, y^1_{d_1 1}, y^1_{12}, \ldots, y^1_{d_1 2}, \ldots, y^k_{d_1 d_2}) \right]$$

We argue that this is equivalent to φ over $\mathcal{B}'$. Indeed, by the surjectivity of I_2, for every element $a_i \in \mathcal{B}$ there are $c^i_1, \ldots, c^i_{d_2} \in \mathcal{C}$ s.t.: $a_i = I_2(c^i_1, \ldots, c^i_{d_2})$, and by the surjectivity of I_1, for every $c^i_j \in \mathcal{C}$ there are $b^i_{1,j}, \ldots, b^i_{d_1,j} \in \mathcal{B}$ s.t. $I_1(b^i_{1,j}, \ldots, b^i_{d_1,j}) = c^i_j$.

So:

$$\mathcal{B} \models \varphi(a_1, \dots, a_k) \Leftrightarrow \mathcal{C} \models \varphi_{I_2}(c_1^1, \dots, c_{d_2}^1, \dots, c_1^k, \dots, c_{d_2}^k)$$

$$\Leftrightarrow \mathcal{B} \models \varphi_{I_2 \circ I_1}(b_{1,1}^1, \dots, b_{d_1,1}^1, b_{1,2}^1, \dots, b_{d_1,2}^1, \dots, b_{d_1 d_2}^k) \qquad \square$$

3.4: Classification Transfer

Let $\mathcal{C}$ be a structure in a finite relational signature. By the classification problem for $\mathcal{C}$, we mean the complexity classification for $CSP(\mathcal{B})$ for all f.o. expansions $\mathcal{B}$ of $\mathcal{C}$.

Lemma 3.4.1: Suppose $\mathcal{D}$ has a pp interpretation I in $\mathcal{C}$, and $\mathcal{C}$ has a pp interpretation J in $\mathcal{D}$ s.t. $J \circ I$ is pp homotopic to $id_{\mathcal{C}}$. Then for every f.o. expansion $\mathcal{C}'$ of $\mathcal{C}$ there is an f.o. expansion $\mathcal{D}'$ of $\mathcal{D}$ s.t. $\mathcal{C}'$ and $\mathcal{D}'$ are mutually pp interpretable.

Proof: Let $\mathcal{C}'$ be a f.o. expansion of $\mathcal{C}$, $c = \dim(I)$, $d = \dim(J)$. Then we set $\mathcal{D}'$ to be the expansion of $\mathcal{D}$ by $\{\varphi_J : \varphi \in \text{signature}(\mathcal{C}') \setminus \text{signature}(\mathcal{C})\}$.

Claim: J is a pp interpretation of $\mathcal{C}'$ in $\mathcal{D}'$

- $\text{dom}(J)$ same as before $\Rightarrow$ still definable;
- $=_J$ same as before;
- R_J same as before for $R \in \text{signature}(\mathcal{C})$;
- $\varphi_J := \varphi_J$ for $\varphi \in \text{signature}(\mathcal{C}') \setminus \text{signature}(\mathcal{C})$.

Claim: I is a pp interpretation of $\mathcal{D}'$ in $\mathcal{C}'$.

- $\text{dom}(I)$ same as before;
- $=_I$ same as before;
- R_I same as before for $R \in \text{signature}(\mathcal{D})$;

The other relation symbols are of the form φ_J of arity dk where φ is a k-ary f.o. definable relation of $\mathcal{C}$. Let $\delta(x_0, x_{11}, \dots, x_{1c}, \dots, x_{d1}, \dots, x_{dc})$ be the p.p. formula that defines

$$J(I(x_{11}, \dots, x_{c1}), \dots, I(x_{1d}, \dots, x_{cd})) = x_0$$

in $\mathcal{C}$. Let:

$$(\varphi_J)_I(x_1^1, \dots, x_d^k) := \exists x^1, \dots, x^k \left[\varphi(x^1, \dots, x^k) \wedge \bigwedge_{i=1}^{k} \delta(x^i, x_{11}^i, \dots, x_{cd}^i) \right]$$

Then:

$$\mathcal{C}' \models (\varphi_J)_I(a_{11}, \ldots, a^k_{cd}) \iff \mathcal{C}' \models \varphi(\alpha^1, \ldots, \alpha^k)$$

where $\alpha^i = J(I(a^i_{11}, \ldots, a^i_{c1}), \ldots, I(a^i_{1d}, \ldots, a^i_{cd}))$

$$\iff \mathcal{D}' \models \varphi_J(I(a^1_{11}, \ldots, a^1_{c1}), \ldots, I(a^k_{1d}, \ldots, a^k_{cd}))$$

as required. $\square$

Consequently, if $\mathcal{C}, \mathcal{D}, \mathcal{C}', \mathcal{D}'$ are as above, and $\mathcal{C}', \mathcal{D}'$ have finite signature then $\mathrm{CSP}(\mathcal{C}') \equiv_p \mathrm{CSP}(\mathcal{D}')$.

<u>Corollary 3.4.2</u>: Let $\mathcal{C}, \mathcal{D}$ be p.p. bi-interpretable. Then every f.o. expansion of $\mathcal{C}$ is pp bi-interpretable with an f.o expansion of $\mathcal{D}$.

<u>Theorem 3.4.3</u>: Let $\mathcal{B}$ be a reduct of Allen's interval algebra that contains $m = \{([u_1, u_2], [v_1, v_2]) : u_2 = v_1\}$. Then $\mathrm{CSP}(\mathcal{B})$ is either in P or NP-complete

<u>Proof</u>: Follows by Example 3.3.3 and the fact (Chapter 12) that all f.o. expansions of $(\mathbb{Q}, <)$ are in P or NP-complete.