RECAP $\mathbb{A}$ a τ-structure $\mathbb{B}$ a σ-structure

A pp-interpretation of dimension d of $\mathbb{B}$ in $\mathbb{A}$ is a partial surjection $I : \mathbb{A}^d \to \mathbb{B}$ s.t.
for every relation R in $\mathbb{B}$ defined by an atomic σ-formula ϕ of arity k,

$$I^{-1}(R) = \{(a_1^1, \dots, a_d^1, \dots, a_1^k, \dots, a_d^k) \mid (I(a_1^1, \dots, a_d^1), \dots, I(a_1^k, \dots, a_d^k)) \in R^{\mathbb{B}}\}$$

has a pp-definition Φ_I in $\mathbb{A}$.

We have a domain formula T_I given by $I^{-1}(T)$ (i.e. $\mathrm{dom}(I)$).

I is **FULL** if

$R \subseteq B^k$ is pp-def in B iff $I^{-1}(R)$ is pp-definable.

$\mathbb{A}$ and $\mathbb{B}$ are **pp-bi-interpretable** if $I : \mathbb{A}^d \to \mathbb{B}$ and $J : \mathbb{B}^k \to \mathbb{A}$ are interpretations AND $I \circ J$ and $J \circ I$ are pp-homotopic
(i.e. $\{I \circ J(\bar{x}) = \mathrm{Id}_B(y)\}$ and $\{J \circ I(\bar{y}) = \mathrm{Id}_A(x)\}$ are pp-def in B and A resp)

§ 3.5 BINARY SIGNATURES & DUAL ENCODING

AIM: $\underbrace{\mathbb{C}}$ is pp-bi-interpretable with $\mathbb{B}$ in a binary signature

A dth full power of $\mathbb{C}$ is a structure $\mathbb{D}$

DOMAIN: $\mathbb{C}^d$ and s.t. $\mathrm{Id}_{\mathbb{C}^d} : \underbrace{\mathbb{C}^d} \to \underbrace{\mathbb{C}^d}_{\mathbb{D}}$ is a full d-dim pp-int. of $\mathbb{D}$ in $\mathbb{C}$

OBS:

$\underbrace{E_{ij}}_{\text{pp-def in } \mathbb{C}^d \text{ + fullness of } \mathrm{Id}_{\mathbb{C}^d}} := \{((x_1, \dots, x_d), (y_1 \dots y_d)) \mid x_i = y_j\}$ is pp-def in $\mathbb{D}$

R pp-def in $\mathbb{C}$ of arity $k \leq d$

$R' := \{(a_1 \dots a_d) \mid (a_1 \dots a_k) \in R\}$ is pp-def in $\mathbb{D}$

pp-bi-int with full powers $\mathbb{D}$ is a dth full power of $\mathbb{C}$, then $\mathbb{D}$ and $\mathbb{C}$ are pp-bi-interpretable.

- $\mathbb{C}$ with maximal arity m. Let $d \geq m$.

$\mathbb{B} = \mathbb{C}^{[d]}$ with domain $\mathbb{C}^d$ and the following relations

BINARY

$E_{ij} := \{((x_1, \dots, x_d), (y_1 \dots y_d)) \mid x_i = y_j\}$

UNARY R'

$R \in \tau$ of arity $k \leq d$ $R' := \{(a_1 \dots a_d) \mid (a_1 \dots a_k) \in R\}$

DUAL ENCODING $\mathbb{C}^{[d]}$ is a full power of $\mathbb{C}$.

- if $\mathbb{C}$ is fin. bdd then so is $\mathbb{C}^{[d]}$
- $Age(\mathbb{C}) = Forb^{emb}(\mathcal{F})$ for $\mathcal{F}$ finite we can compute in poly time wrt $|\mathcal{F}|$ $\mathcal{F}'$ s.t. $Age(\mathbb{C}^{[d]}) = Forb^{emb}(\mathcal{F}')$.

Proof:

$Id : \mathbb{C}^d \to \mathbb{C}^d$ is a pp-int of dim d of $\mathbb{C}^{[d]}$ in $\mathbb{C}$.

$\pi : \mathbb{C}^d \to \mathbb{C}$ is a 1-dim pp-int. of $\mathbb{C}$ in $\mathbb{C}^{[d]}$.

by Lemma 2.4.8 it is sufficient to prove Id and π give a pp-bi-int to deduce fullness.

so any struc. $\mathbb{C}$ of max arity m is pp-bi-int with a binary $\mathbb{B}$ ☒

§ 3.6 pp-constructions

$\mathcal{C}$ a class of structures

$H(\mathcal{C})$ structs hom eq to $\mathbb{C} \in \mathcal{C}$.

$C(\mathcal{C})$ structures obtained by expanding $\mathbb{B} \in \mathcal{C}$ by fin many singleton rels isolating $b \in B$ whose $\mathrm{Aut}(\mathbb{B})$-orbit is pp-def in $\mathbb{B}$.

$P^{\mathrm{fin}}_{\mathrm{full}}(\mathcal{C})$ full finite powers of $\mathbb{C} \in \mathcal{C}$

$\mathrm{Red}(\mathcal{C})$ pp-reducts of $\mathbb{C} \in \mathcal{C}$

$I(\mathcal{C})$ pp-interpretable from structures from $\mathcal{C}$.

<u>Obs</u>: - $\mathrm{Red}(P^{\mathrm{fin}}_{\mathrm{full}}(\mathcal{C})) \subseteq I(\mathcal{C})$

- $I(I(\mathcal{C})) = I(\mathcal{C})$

- $C(C(\mathcal{C})) = C(\mathcal{C})$

BARTO, OPRŠAL, PINSKER $\mathcal{C}$ a class of structures.

Let $\mathcal{D}$ be the smallest class containing $\mathcal{C}$ and closed under H, C and I.

$$\mathcal{D} = \mathrm{H}\,\mathrm{Red}\,\mathrm{P}^{fin}_{full}(\mathcal{C}) = \mathrm{H}\,\mathrm{I}(\mathcal{C})$$

If $\mathbb{A} \in \mathrm{H}\,\mathrm{Red}\,\mathrm{P}^{fin}_{full}(\mathbb{B})$ we say $\mathbb{A}$ is pp-constructible in $\mathbb{B}$.

$(\mathbb{B}, c) \in \mathrm{H}\,\mathrm{I}(\mathbb{B})$ $\mathbb{B}$ $c \in B$ s.t. Aut$(\mathbb{B})$-orbit of c is pp-def.

$$\mathbb{C} := (\mathbb{B}, \underset{S = \{c\}}{c}) \in \mathrm{H}\,\mathrm{I}(\mathbb{B})$$

Proof: O the orbit of c under Aut$(\mathbb{B})$ ϕ is the pp-def.

We give a 2-dim pp-int in $\mathbb{B}$ I of a structure $\mathbb{A}$ with same lang as $\mathbb{C}$ and domain $B \times O$

$I : B^2 \to \underbrace{B \times O}_{A}$ DOMAIN is $B \times O$ and I is just the identity on $B \times O$

(*) $R^A := \{((a_1, b_1), \dots, (a_k, b_k)) \in B \times O \mid (a_1 \dots a_k) \in R^B \; b_1 = \dots = b_k \in O\}$

(A) $S^A := \{(a,a) \mid a \in O\}$

(∗) $R^{\mathbb{A}} := \{ ((a_1, b_1), \dots, (a_k, b_k)) \in B \times O \mid (a_1 \dots a_k) \in R^{\mathbb{B}} \quad b_1 = \dots = b_k \in O \}$

(△) $S^{\mathbb{A}} := \{ (a, a) \mid a \in O \}$

<u>CLAIM</u>: $\mathbb{A}$ and $\mathbb{C}$ are hom equiv.

$g : \mathbb{C} \to \mathbb{A} \qquad a \mapsto (a, c)$

$\bar{a} \in R^{\mathbb{C}} = R^{\mathbb{B}} \overset{(*)}{\Longrightarrow} ((a_1, c), \dots, (a_k, c)) \in R^{\mathbb{A}}$

$S^{\mathbb{C}} = \{c\}$ by (△) $(c, c) \in S^{\mathbb{A}}$

$b \in O \qquad \alpha_b \in \mathrm{Aut}(\mathbb{B})$ s.t. $\alpha_b(b) = c$. Set $h(a, b) = \alpha_b(a)$

$h : \mathbb{A} \to \mathbb{C}$

$\bar{t}\ ((a_1, b), \dots, (a_k, b)) \in R^{\mathbb{A}}$. $h(\bar{t}) = \underbrace{(\alpha_b(a_1), \dots, \alpha_b(a_k))}_{\in R^{\mathbb{C}}}$

$(a_1 \dots a_k) \in R^{\mathbb{B}} = R^{\mathbb{C}}$ and α_b preserves $R^{\mathbb{B}}$

<u>S is pres</u>: For $a \in O \quad S^{\mathbb{A}}(a, a)$

$h(a, a) = \alpha_a(a) = c \in \{c\} = S^{\mathbb{C}}$

∎

$I(\mathbb{B}) \subseteq H\,Red\,P^{fin}_{full}(\mathbb{B})$

<u>Proof</u>: $\mathbb{C}$ with a d-dim pp-int I in $\mathbb{B}$

Take $\mathbb{D}$ a dth full pw of $\mathbb{B}$.

Define $\mathbb{D}'$ on $\mathbb{B}^d$ as follows:

For $R \in \tau$ $\quad I^{-1}(R)$ to be its int. in $\mathbb{D}'$

$\mathbb{D}' \in Red(\mathbb{D})$:

$$\begin{array}{ccc} \mathbb{B}^d & \xrightarrow{I} & \mathbb{C} \\ & \xleftarrow{\quad} & R \text{ pp-def} \\ {\scriptstyle Id}\downarrow & & \\ & & \mathbb{B}^d = \mathbb{D} \end{array}$$

$I^{-1}(R)$ is pp-def

$I^{-1}(R)$ is pp-def in $\mathbb{D}$
<u>because</u> Id is <u>FULL</u> int.

<u>CLAIM</u>: $\mathbb{D}'$ is hom eq. to $\mathbb{C}$

$f : \mathbb{D}' \longrightarrow \mathbb{C}$ extending I ✓

$\mathbb{D}' = \mathbb{B}^d$

$g : \mathbb{C} \longrightarrow \mathbb{D}'$ s.t. $f \circ g = Id_{\mathbb{C}}$ ✓

∎

MORE USEFUL CORRESPONDENCES

⓪ $(\mathbb{B}, c) \in H\,I(\mathbb{B})$

① $I(\mathbb{B}) \subseteq H\,Red\,P^{fin}_{full}(\mathbb{B})$

② $H\,H(\mathcal{C}) = H(\mathcal{C})$

③ $Red\,Red(\mathcal{C}) = Red(\mathcal{C})$

④ $P^{fin}_{full}\,Red(\mathcal{C}) \subseteq Red\,P^{fin}_{full}(\mathcal{C})$

⑤ $H\,Red\,H\,Red(\mathcal{C}) = H\,Red(\mathcal{C})$

⑥ $P^{fin}_{full}\,H(\mathcal{C}) \subseteq H\,Red\,P^{fin}_{full}(\mathcal{C})$

⑦ $P^{fin}_{full}\,P^{fin}_{full}(\mathcal{C}) = P^{fin}_{full}(\mathcal{C})$

$$\mathcal{D} = H\,Red\,P^{fin}_{full}(\mathcal{C}) = HI(\mathcal{C})$$

Proof:

$$\underbrace{H\,Red\,P^{fin}_{full}(\mathcal{C})}_{\text{closed under } H,\ C \text{ and } I} \overset{\checkmark}{\subseteq} HI(\mathcal{C}) \overset{\checkmark}{\subseteq} \mathcal{D}$$

$\mathcal{D} \subseteq \dots$

MORE USEFUL CORRESPONDENCES

0. $(\mathbb{B}, c) \in HI(\mathbb{B})$
1. $I(\mathbb{B}) \subseteq H\,Red\,P^{fin}_{full}(\mathbb{B})$
2. $H\,H(\mathcal{C}) = H(\mathcal{C})$
3. $Red\,Red(\mathcal{C}) = Red(\mathcal{C})$
4. $P^{fin}_{full}\,Red(\mathcal{C}) \subseteq Red\,P^{fin}_{full}(\mathcal{C})$
5. $H\,Red\,H\,Red(\mathcal{C}) = H\,Red(\mathcal{C})$
6. $P^{fin}_{full}\,H(\mathcal{C}) \subseteq H\,Red\,P^{fin}_{full}(\mathcal{C})$
7. $P^{fin}_{full}\,P^{fin}_{full}(\mathcal{C}) = P^{fin}_{full}(\mathcal{C})$

CLOSURE UNDER I:

$$I(H\,Red\,P^{fin}_{full}(\mathcal{C})) \overset{1}{\subseteq} H\,Red\,P^{fin}_{full}\,H\,Red\,P^{fin}_{full}(\mathcal{C})$$

$$\overset{6}{\subseteq} H\,Red\,H\,Red\,P^{fin}_{full}\,Red\,P^{fin}_{full}(\mathcal{C}) \overset{4+5+6}{\subseteq} \underbrace{H\,Red\,H\,Red}_{5}\,P^{fin}_{full} = H\,Red\,P^{fin}_{full}(\mathcal{C})$$

SWAP

CLOSURE UNDER C:

$$C\,H\,Red\,P^{fin}_{full}(\mathcal{C}) \overset{0}{\subseteq} H\,I\,\underbrace{H\,Red\,P^{fin}_{full}(\mathcal{C})} \subseteq H\,H\,Red\,P^{fin}_{full}(\mathcal{C}) \subseteq H\,Red\,P^{fin}_{full}$$

✓

$\boxtimes$

$K_3 \in HI(\mathbb{B}) \Rightarrow \mathbb{B}$ has a fn. sign. reduct which is NP-hard

CONJ: $K_3 \notin HI(\mathbb{B}) \Rightarrow CSP(\mathbb{B}) \in P$.