COUNTABLE CATEGORICITY

1. Introducing countable categoricity

Definition 1. Let T be a first-order theory. We say that T is *ω-categorical* if all models of T of cardinality $\aleph_0$ are isomorphic. For a first-order structure $\mathbb{B}$ we shall say that $\mathbb{B}$ is *ω-categorical* if $\mathsf{Th}(\mathbb{B})$ is ω-categorical.

Remark 2. Most model theoretic literature defines ω-categoricity, slightly differently. More precisely, the usual definition of ω-categoricity requires the *existence* of a unique, up to isomorphism, model of cardinality $\aleph_0$. The difference here is that usually finite structures are not considered ω-categorical – here they will be!

Examples. The following are standard examples of ω-categorical structures/theories: $(\mathbb{Q}, <)$; RG (the *random graph*); the theory of *atomless Boolean algebras.*

1.1. The theorem of Engeler, Svenonius, Ryll-Nardzewski.

In this section, we will prove the most fundamental theorem about ω-categorical structures. First, some terminology:

Definition 3. Let X be a set. A set of permutations of X is called a *permutation group on X* if it contains the identity and is closed under composition and inverses (i.e. if it is a group).

Definition 4. Let G be a permutation group on X and $\bar{t} = (t_1, \ldots, t_n) \in X^n$ an n-tuple from X. The *orbit* of $\bar{t}$ is the set $\{(\alpha t_1, \ldots, \alpha t_n) : \alpha \in G\}$.

Definition 5. A permutation group on X is called *oligomorphic* if it has finitely many orbits on n-tuples for all $n \in \mathbb{N}$.

Theorem 6 (Engeler, Svenonius, Ryll-Nardzewski)**.** *Let $\mathbb{B}$ be a countable structure in a countable signature. Then, the following are equivalent:*

(1) $\mathbb{B}$ is ω-categorical.
(2) All types of $\mathbb{B}$ are principal.
(3) All models of $\mathsf{Th}(\mathbb{B})$ are atomic.
(4) For all $n \in \mathbb{N}$ every set of n-tuples that is preserved under all automorphisms of $\mathbb{B}$ is definable in $\mathbb{B}$.
(5) The automorphism group of $\mathbb{B}$ is oligomorphic.
(6) For each $n \in \mathbb{N}$ there are finitely many inequivalent over $\mathbb{B}$ formulas in n free variables.
(7) $\mathbb{B}$ has finitely many n-types for all $n \in \mathbb{N}$.

Proof.

(1) $\Rightarrow$ (2): If $\mathbb{B}$ has a non-principal type, then there is a countable model of $\mathsf{Th}(\mathbb{B})$ realising it and one omitting it.

(2) $\Rightarrow$ (3): This is by definition. (If $\mathbb{A} \vDash \mathsf{Th}(\mathbb{B})$ then every element of $\mathbb{A}$ realises a type of $\mathbb{B}$, which is principal, so $\mathbb{A}$ is atomic).

(3) $\Rightarrow$ (1): Any two countable atomic structures with the same theory are isomorphic.

Date: Notes written up by Aris Papadopoulos, following Section 4.1 of [Bod21]. All mistakes are, of course, due to him.

(2) ⇒ (4): First, by (2) we have that $\mathbb{B}$ is itself atomic. In particular, any two n-tuples with the same type are in the same orbit of $\mathsf{Aut}(\mathbb{B})$ (and, of course, any two elements in the same orbit have the same type) so types determine the orbits of $\mathsf{Aut}(\mathbb{B})$. Since types are isolated, the orbits of $\mathsf{Aut}(\mathbb{B})$, which are precisely the sets preserved by all automorphisms, are definable by the isolating formulas.

(4) ⇒ (5): Suppose that $\mathsf{Aut}(\mathbb{B})$ is not oligomorphic. Then, for some $n \in \mathbb{N}$ there are infinitely many orbits of n-tuples. Of course, orbits are preserved by all automorphisms (by definition) and thus by (4) each of these sets is definable. But then, any subset of this collection will be definable, giving us uncountably many definable subsets of n-tuples, which cannot happen if the language is countable.

(5) ⇒ (6): If for some $n \in \mathbb{N}$ there were infinitely many inequivalent formulas over $\mathbb{B}$, then $Aut(\mathbb{B})$ would have infinitely many orbits on n-tuples (since automorphisms preserve first-order formulas).

(6) ⇒ (7): Since there are only finitely many inequivalent formulas over $\mathbb{B}$ in n-free variables we can only build finitely many n-types of $\mathbb{B}$.

(7) ⇒ (2): If $\mathbb{B}$ has finitely many n-types, then, for each pair we may pick a formula separating them (i.e. belonging to one and not to the other). Then types are isolated by Boolean combinations of the formulas separating them.

□

1.2. Compactness and ω-categoricity.

Lemma 7. *Let $\mathbb{B}$ be a finite or countable ω-categorical structure and $\mathbb{A}$ a countable structure. If there is no homomorphism (resp. embedding) of $\mathbb{A}$ into $\mathbb{B}$ then there is some finite substructure of $\mathbb{A}$ which does not homomorphically map (resp. embed) into $\mathbb{B}$.*

Proof. We shall prove the contrapositive. So suppose that every finite substructure of $\mathbb{A}$ embeds into $\mathbb{B}$. The goal is to build an embedding of $\mathbb{A}$ into $\mathbb{B}$. To this end, fix an enumeration of $\mathbb{A}$, say $\{a_1, a_2, \dots\}$. For each $n \in \mathbb{N}$ define an equivalence relation $\sim_n$ on the set of all embeddings $\{a_1, \dots, a_n\} \to \mathbb{B}$ as follows:

$$f \sim_n g \text{ if, and only if, } \exists \alpha \in \mathsf{Aut}(\mathbb{B}) \text{ s.t. } \alpha f = g.$$

We now build a tree:

- On the n-th level the nodes shall be the equivalence classes of $\sim_n$.
- A node on $[f]_{\sim_n}$ on level n will be a parent of a node $[g]_{\sim_{n+1}}$ if there are $\tilde{f} \in [f]_{\sim_n}$ and $\tilde{g} \in [g]_{\sim_{n+1}}$ such that $\tilde{g} \restriction_{\{a_1,\dots,a_n\}} = \tilde{f}$.

We now observe the following:

(A) By Theorem 6, this tree is finitely branching (as there are only finitely many orbits of $\mathsf{Aut}(\mathbb{B})$, and each equivalence class of $\sim_n$ must belong to some orbit).

(B) By assumption, all levels of the tree have nodes.

Thus, by König's lemma there is an infinite path down this tree, that is, a path from the root which passes through each level, say $\{[f_i]_{\sim_i} : i \in \mathbb{N}\}$. We shall use this path to inductively construct an embedding $e : \mathbb{A} \to \mathbb{B}$, by taking the restriction of e to $\{a_1, \dots, a_n\}$ to be an element from the n-th node of the path. More precisely, suppose that that e has been defined on $\{a_1, \dots, a_n\}$. By construction of the infinite path, there are representatives $\tilde{e}_n \in [f_n]_{\sim_n}$ and $\tilde{e}_{n+1} \in [f_{n+1}]_{\sim_{n+1}}$, and by inductive hypothesis there is some $\alpha \in \mathsf{Aut}(\mathbb{B})$ such that $e = \alpha\tilde{e}_n$. Let $e(a_{n+1}) = \alpha\tilde{e}_{n+1}(a_{n+1})$. Clearly the restriction of e to $\{a_1, \dots, a_{n+1}\}$ is an element of $[f_{n+1}]_{\sim_{n+1}}$. It is immediate from the definition that $e : \mathbb{A} \to \mathbb{B}$ is indeed an embedding. □

We give two corollaries:

Corollary 8. *Let $\mathbb{C}$ be any structure. Then, the following are equivalent:*

(1) There is a finite structure $\mathbb{B}$ with the same CSP *as $\mathbb{C}$.*
(2) $\mathbb{C}$ has a finite core.

Proof. (2) $\Rightarrow$ (1) is trivial (since if $\mathbb{C}$ has a finite core then $\mathbb{B}'$ then $\mathbb{B}$ and $\mathbb{C}$ are homomorphically equivalent). For (1) $\Rightarrow$ (2), suppose that $\mathsf{CSP}(\mathbb{B}) = \mathsf{CSP}(\mathbb{C})$ for some finite structure $\mathbb{B}$. Since all finite structures have a (unique) core, let $\mathbb{B}'$ be the core of $\mathbb{B}$. Then, all finite substructures of $\mathbb{C}$ homomorphically map into $\mathbb{B}'$ (since $\mathbb{B}$ and $\mathbb{B}'$ are homomorphically equivalent) then by the previous lemma $\mathbb{C}$ homomorphically maps into $\mathbb{B}'$. Clearly $\mathbb{B}'$ homomorphically maps into $\mathbb{C}$, so $\mathbb{B}'$ is a finite core of $\mathbb{C}$. □

Corollary 9. *Let $\mathbb{A}$ and $\mathbb{B}$ be countable ω-categorical structures. Then, the following are equivalent:*

(1) $\mathsf{CSP}(\mathbb{A}) = \mathsf{CSP}(\mathbb{B})$.
(2) There is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$ and a homomorphism from $\mathbb{B}$ to $\mathbb{A}$.

Proof. Immediate from the previous lemma. □

WARNING This is not the case for general countable relational structures (e.g. the CSP of the infinite line $(\mathbb{Z}; \{(x,y) : y = x+1\})$ is equal to the CSP of the infinite ray $(\mathbb{N}; \{(x,y) : y = x+1\})$, but the infinite line does not homomorphically map into the infinite ray).

We give a stronger version of the previous lemma:

Proposition 10. *Let $\mathbb{B}$ be a countable ω-categorical structure in a countable relational signature τ and $\mathbb{A}$ any (countably infinite) τ-structure. Let σ be a set of function symbols. Then, for any universal $(\tau \cup \sigma)$-theory T the following are equivalent:*

(1) The two sorted τ-structure $(\mathbb{A}, \mathbb{B})$ has a $(\tau \cup \sigma)$-expansion that satisfies T such that every $f \in \sigma$ denotes a function from $\mathbb{A}$ to $\mathbb{B}$.
(2) For every finite induced substructure $\mathbb{C}$ of $\mathbb{A}$ the two-sorted τ-structure $(\mathbb{C}, \mathbb{B})$ has a $(\tau \cup \sigma)$-expansion that satisfies T such that every $f \in \sigma$ denotes a function from $\mathbb{C}$ to $\mathbb{B}$.

Proof. The implication (1) $\Rightarrow$ (2) is trivial, since T is universal. For the other implication I give a sketch:

- Let P_1, P_2 be two new unary relation symbols.
- Let D be the quantifier-free theory of $\mathbb{A}$ expanded by constants naming each element.
- Let S be a set of sentences expressing that:
  (1) P_1 and P_2 are disjoint and denote two distinct sorts such that all function symbols in σ are functions from P_1 to P_2.
  (2) The τ-reduct of the structure induced by the elements of P_2 has the same first-order theory as $\mathbb{B}$.
- By (2) and compactness $T \cup D \cup S$ is satisfiable.
- By Löwenheim-Skolem there is a countable model of $T \cup D \cup S$.
- The substructure of the countable model generated by the constants and the elements named by P_2 must be isomorphic to $(\mathbb{A}, \mathbb{B})$, since we have named all the elements of $\mathbb{A}$ and $\mathbb{B}$ is ω-categorical.

□

References

[Bod21] Manuel Bodirsky. *Complexity of Infinite-Domain Constraint Satisfaction*. Cambridge University Press, May 2021.

Appendix A. Background definitions

I'll gather here some of the relevant background, to make sure we're on the same page:

Definition A.1. A type $p(\bar{x})$ of a theory T is called *principal* (sometimes *isolated*) if there is a formula $\phi(\bar{x})$ such that:

(1) $T \cup \{(\exists \bar{x})\phi(\bar{x})\}$ is satisfiable.
(2) For all $\psi(\bar{x}) \in p(\bar{x})$ we have that $T \vDash (\forall \bar{x})(\phi(\bar{x}) \to \psi(\bar{x}))$.

In this case, we say that $\phi(\bar{x})$ *isolates* $p(\bar{x})$.

Definition A.2. Let $\mathbb{B}$ be a structure. An n-type $p(\bar{x})$ of $\mathsf{Th}(\mathbb{B})$ is called *realised* in $\mathbb{B}$ if there is some $\bar{a} \in B^n$ such that $\mathbb{B} \vDash \phi(\bar{a})$ for all $\phi(\bar{x}) \in p(\bar{x})$. An n-type of $\mathsf{Th}(\mathbb{B})$ which is not realised in $\mathbb{B}$ is called *omitted*.

Theorem A.3. *Let T be a theory in a countable signature and $\Sigma = \{p_i(\bar{x}) : i \in \omega\}$ a set of non-principal types of T. Then, there is a countable model $\mathbb{B} \vDash T$ in which all types in Σ are omitted.*

Definition A.4. A structure $\mathbb{B}$ is called *atomic* if all types realised in $\mathbb{B}$ are principal, i.e. for all $\mathsf{tp}(\bar{a})$ is principal for all $\bar{a} \in B^n$.

Theorem A.5. *Any two countable atomic structures with the same theory are isomorphic. In particular, if $\mathbb{B}$ is a countable atomic structure and $\bar{a}, \bar{b} \in B^n$ have the same type, then they are in the same orbit of $\mathsf{Aut}(\mathbb{B})$.*

Definition A.6. A structure $\mathbb{B}$ is called a *core* if all of its endomorphisms are embeddings. A *core* of $\mathbb{B}$ is a structure $\mathbb{A}$ which is a core and is homomorphically equivalent to $\mathbb{B}$.

Theorem A.7. *All finite structures have a core, which is unique up to isomorphism.*