COUNTABLE CATEGORICITY

2. Oligomorphic permutation groups

2.1. **Topology.** Let $\mathbb{B}$ be a relational structure. We define $\mathsf{Sym}(B)$ to be the set of all permutations of the domain B of $\mathbb{B}$.

Definition 1. We say that $\mathcal{P} \subseteq \mathsf{Sym}(\mathbb{B})$ is *closed* if the following condition holds:

> For all $\alpha \in \mathsf{Sym}(B)$ if for every finite $A \subseteq B$ there is some $\beta \in \mathcal{P}$ such that for all $x \in A$ we have that $\alpha(x) = \beta(x)$ then $\alpha \in \mathcal{P}$.

Remark 2. Let us say that $\mathcal{Q} \subseteq \mathsf{Sym}(B)$ is *open* if its complement is closed. Then, by unfolding the definitions, we get that $\mathcal{Q}$ is open if, and only if, it is a union of sets of the form:

$$\{\alpha \in \mathsf{Sym}(B) : (\forall x \in A)\alpha(x) = \beta(x)\},$$

for some fixed $\beta \in \mathsf{Sym}(B)$ and finite set $A \subseteq B$ (a set of this form is the *coset of the stabilisers of a finite tuple*).

Definition 3. Let $\mathcal{P} \subseteq \mathsf{Sym}(B)$. We define the *strong invariants* of $\mathcal{P}$, denoted $\mathsf{sInv}(\mathcal{P})$ to be the set of all relations R on B such that for all $\alpha \in \mathcal{P}$ both α and α^{-1} preserve R.

Proposition 4. *Let $\mathcal{P} \subseteq \mathsf{Sym}(B)$. Then, the following are equivalent:*

(1) $\mathcal{P}$ is the automorphism group of some relational structure with domain B.
(2) $\mathcal{P}$ is a closed subgroup of $\mathsf{Sym}(B)$.
(3) $\mathcal{P}$ is the automorphism group of a homogeneous relational structure with domain B

Proof.

(1) ⇒ (2): Of course, if $\mathcal{P}$ is an automorphism group it is a group, so we need only show that it is closed. Suppose that $\alpha \in \mathsf{Sym}(B)$ is such that for every finite $A \subseteq B$ there is some $\beta \in \mathcal{P}$ such that $\alpha x = \beta x$ for all $x \in A$. Then, α must be itself an automorphism, for otherwise this would be witnessed from the restriction of α to a finite set.

(2) ⇒ (3): We claim that if $\mathcal{P}$ is a closed subgroup of $\mathsf{Sym}(B)$ then $\mathsf{sInv}(\mathcal{P})$ is homogeneous and $\mathcal{P} = \mathsf{Aut}(\mathsf{sInv}(\mathcal{P}))$. First, we obviously have that $\mathcal{P} \subseteq \mathsf{Aut}(\mathsf{sInv}(\mathcal{P}))$. Now, suppose that $\alpha \in \mathsf{Aut}(\mathsf{sInv}(\mathcal{P}))$. Then, for all finite $\{a_1, \dots, a_n\} \subseteq B$ consider the relation:

$$R_{a_1,\dots,a_n} = \{(\beta a_1, \dots, \beta a_n) : \beta \in \mathcal{P}\}.$$

Clearly, R is in $\mathsf{sInv}(\mathcal{P})$, since $\mathcal{P}$ is a group, and of course, since α is an automorphism of $\mathsf{sInv}(\mathcal{P})$ then it must preserve this set. In particular, there is some $\beta \in \mathcal{P}$ such that $\alpha a_i = \beta a_i$ for all $i = 1, \dots, n$. Since $\mathcal{P}$ is closed, we must have that $\alpha \in \mathcal{P}$. We actually have also essentially showed homogeneity, since, any finite isomorphism, say with domain $\{a_1, \dots, a_n\}$ preserves the relation $R_{a_1,\dots,a_n}$, and thus (again by closure) this finite isomorphism is the restriction of an element in $\mathcal{P}$.

(3) ⇒ (1): Trivial.

□

Date: 16/10/23 – Notes written up by Aris Papadopoulos, following Section 4.2 of [**?**]. All mistakes are, of course, due to him.

Remark 5. We have seen that being ω-categorical is a property of the automorphism group. Thus, in fact by the previous proposition we have that every ω-categorical structure $\mathbb{B}$ has the same automorphism group as a homogeneous ω-categorical structure, which, by the proof, is just $\mathsf{sInv}(\mathsf{Aut}(\mathbb{B}))$.

2.2. The sInv-Aut Galois connection.

Definition 6. An *(anti-tone) Galois connection* is a pair of functions:

$$F : U \to V, \text{ and } G : V \to U,$$

between two posets F and G such that:

$$v \leq F(u) \text{ if, and only if, } u \leq G(v),$$

for all $u \in U$ and $v \in V$.

Remark 7. For all $u \in U$ and $v \in V$ we have the following:

- $u \leq G(F(u))$ [Why? Apply the definition to $F(u) \leq F(u)$].
- $v \leq F(G(v))$
- $F(u) = F(G(F(u)))$ [Why? We have already shown $\leq$, for $\geq$ apply the definition to $u \leq G(F(u)) \leq G(F(G(F(u))))$]
- $G(v) = G(F(G(v))$

Proposition 8. *Let B be a set. The operations* Aut *and* sInv *form a Galois connection between the set of all relations over B and the sets of permutations of B.*

Proof. Let $\mathcal{R}$ be a set of relations over B and $\mathcal{P}$ a set of permutations of B. We have to show that:

$$\mathcal{P} \subseteq \mathsf{Aut}(\mathcal{R}) \text{ if, and only if, } \mathcal{R} \subseteq \mathsf{sInv}(\mathcal{P}).$$

$\Rightarrow$: If $\mathcal{P} \subseteq \mathsf{Aut}(\mathcal{R})$ then for all $R \in \mathcal{R}$ and $g \in \mathcal{P}$ we have that both g and g^{-1} preserve R.
$\Leftarrow$: If $\mathcal{R} \subseteq \mathsf{sInv}(\mathcal{P})$ then for all $g \in \mathcal{P}$ both g and g^{-1} preserve R, so $g \in \mathsf{Aut}(\mathcal{R})$.

□

Definition 9. Let $\mathcal{P} \subseteq \mathsf{Sym}(B)$. We define the following:

- The *permutation group generated by* $\mathcal{P}$, denoted $\langle \mathcal{P} \rangle$, to be the smallest permutation group containing $\mathcal{P}$.
- The *closure of* $\mathcal{P}$ *in* $\mathsf{Sym}(B)$, denoted $\overline{\mathcal{P}}$, to be the smallest closed subset of $\mathsf{Sym})B)$ containing $\mathcal{P}$.

Remark 10. Explicitly, $\overline{\mathcal{P}}$ contains $\mathcal{P}$ together with all the permutations $\alpha \in \mathsf{Sym}(B)$ such that for all finite $A \subseteq B$ there is some $\beta \in \mathcal{P}$ such that $\alpha(x) = \beta(x)$ for all $x \in A$ (i.e. it is exactly $\mathcal{P}$ together with its "limit points").

Proposition 11. *Let $\mathcal{P} \subseteq \mathsf{Sym}(B)$, and define $\mathcal{P}^\star$ to be the smallest permutation group that contains $\mathcal{P}$ and is closed in* $\mathsf{Sym}(B)$*. Then:*

$$\mathcal{P}^\star = \overline{\langle \mathcal{P} \rangle} = \mathsf{Aut}(\mathsf{sInv}(\mathcal{P})).$$

Proof.

- First we shall show that $\mathcal{P}* = \overline{\langle \mathcal{P} \rangle}$ Note that $\mathcal{P} \subseteq \mathcal{P}^\star$ and since $\mathcal{P}^\star$ is a permutation group we immediately get (by minimality) that $\langle \mathcal{P} \rangle \subseteq \mathcal{P}^\star$. Now, since $\mathcal{P}^\star$ is closed, this (again by minimality) implies that $\overline{\langle \mathcal{P} \rangle} \subseteq \mathcal{P}^\star$. Let us then show the converse. Since $\mathcal{P}^\star$ is the smallest permutation group that contains $\mathcal{P}$ and is closed in $\mathsf{Sym}(B)$ (and $\overline{\langle \mathcal{P} \rangle}$ contains $\mathcal{P}$) it suffices to show that $\overline{\langle \mathcal{P} \rangle}$ is a permutation group (we already know that it is closed). Showing that it contains the identity is immediate (since $\langle\mathcal{P}\rangle$ is a permutation group), so it suffices to show closure under inverses and composition. Let us only do the former (the argument for the latter is similar). Suppose that $\alpha, \beta \in \overline{\langle\mathcal{P}\rangle}$. By definition of $\overline{\langle\mathcal{P}\rangle}$ we must have that for all finite $A \subseteq \beta$ there are $\alpha', \beta' \in \langle\mathcal{P}\rangle$ such that $\alpha x = \alpha' x$ and $\beta x = \beta' x$ for all $x \in A$. Since $\langle\mathcal{P}\rangle$ is a permutation group it contains their composition, and hence, since $\overline{\langle\mathcal{P}\rangle}$ is closed it must contain $\alpha\beta$.

- Now we shall show that $\overline{\langle\mathcal{P}\rangle} = \mathsf{Aut}(\mathsf{sInv}(\mathcal{P}))$. On the one hand, if $\alpha \in \overline{\langle\mathcal{P}\rangle}$, then we claim that both α and α^{-1} preserve each $R \in \mathsf{sInv}(\mathcal{P})$. Indeed, suppose that $R \in \mathsf{sInv}(\mathcal{P})$ and $t \in R$. By definition of $\overline{\langle\mathcal{P}\rangle}$ there are $\beta_1, \ldots, \beta_k \in \mathcal{P} \cup \mathcal{P}^{-1}$ such that $\alpha t = (\beta_1 \circ \cdots \circ \beta_k)t$. Since each β_i preserves R it follows that so does α (and α^{-1}, similarly). Conversely, if $\alpha \in \mathsf{Aut}(\mathsf{sInv}(\mathcal{P}))$ then α and its inverse preserve the relation:
$$\{(\beta t_1, \ldots, \beta t_n) : \beta \in \langle\mathcal{P}\rangle\},$$
and by closure we must have that $\alpha \in \overline{\langle\mathcal{P}\rangle}$.

□

Proposition 12. *Let* $\mathbb{B}$ *be any structure. Then:*
$$\langle\mathbb{B}\rangle_{\mathsf{fo}} \subseteq \mathsf{sInv}(\mathsf{Aut}(\mathbb{B})),$$
where $\langle\mathbb{B}\rangle_{\mathsf{fo}}$ *denotes the set of all first-order definable relations in* $\mathbb{B}$*.*

Proof. If $R \in \langle\mathbb{B}\rangle_{\mathsf{fo}}$ then all $g, g^{-1} \in \mathsf{Aut}(\mathbb{B})$ preserve R. □

The following is an immediate consequence of Ryll-Nardzewski:

Theorem 13. *If* $\mathbb{B}$ *is* ω*-categorical, in a countable signature, then:*
$$\mathsf{sInv}(\mathsf{Aut}(\mathbb{B})) = \langle\mathbb{B}\rangle_{\mathsf{fo}},$$
and this characterises ω*-categorical structures.*

Remark 14. So, $\mathsf{Aut}(\mathbb{B})$ is precisely the automorphism group of $\mathsf{sInv}(\mathsf{Aut}(\mathbb{B}))$, which in the ω-categorical case is precisely $\langle\mathbb{B}\rangle_{\mathsf{fo}}$. Thus, if $\mathbb{B}$ is ω-categorical we have that:
$$\mathsf{Aut}(\mathbb{B}) = \mathsf{Aut}(\langle\mathbb{B}\rangle_{\mathsf{fo}}).$$

Theorem 15. *Let* $\mathbb{B}$ *be a countable* ω*-categorical structure. Then:*

- *The sets of the form* $\langle\mathbb{A}\rangle_{\mathsf{fo}}$ *where* $\mathbb{A}$ *is a first-order reduct of* $\mathbb{B}$ *ordered by inclusion form a lattice.*
- *The closed supergroups of* $\mathsf{Aut}(\mathbb{B})$ *in* $\mathsf{Sym}(B)$ *ordered by inclusion form a lattice.*
- *The operator* sInv *is an anti-isomorphism between these two lattices and* Aut *is its inverse.*

References

[Bod21] Manuel Bodirsky. *Complexity of Infinite-Domain Constraint Satisfaction.* Cambridge University Press, May 2021.