COUNTABLE CATEGORICITY

2. Oligomorphic permutation groups (ctd.)

Last time we left off after stating the following theorem:

Theorem 1. *Let $\mathbb{B}$ be a countable ω-categorical structure. Then:*

- *The sets of the form $\langle \mathbb{A} \rangle_{\mathsf{fo}}$ where $\mathbb{A}$ is a first-order reduct of $\mathbb{B}$ ordered by inclusion form a lattice.*
- *The closed supergroups of $\mathsf{Aut}(\mathbb{B})$ in $\mathsf{Sym}(B)$ ordered by inclusion form a lattice.*
- *The operator sInv is an anti-isomorphism between these two lattices and Aut is its inverse.*

One can prove this result by putting together the results that we stated last time. Here is a more self-contained proof.

Proof. The first two items are almost immediate. We prove the third:

(i) Aut is surjective: Indeed, if $\mathcal{P} \subseteq \mathsf{Sym}(B)$ is closed and contains $\mathsf{Aut}(\mathbb{B})$ then by Ryll-Nardzewski $\mathsf{sInv}(\mathcal{P})$ is a reduct of $\mathbb{B}$ and $\mathsf{Aut}(\mathsf{sInv}(\mathcal{P})) = \mathcal{P}$ (since $\mathcal{P} = \overline{\langle \mathcal{P} \rangle}$.

(ii) Aut is injective: If $\mathsf{Aut}(\mathsf{sInv}(\langle \mathbb{A} \rangle_{\mathsf{fo}})) = \mathsf{Aut}(\mathsf{sInv}(\langle \mathbb{A}' \rangle_{\mathsf{fo}}))$ then the strong invariants of these structures are equal, and thus the first-order definable sets in $\mathbb{A}$ and $\mathbb{A}'$ are equal.

□

Remark 2. Let $\mathbb{A}$ and $\mathbb{B}$ be ω-categorical. Then, the following are equivalent:

(1) $\mathbb{A}$ and $\mathbb{B}$ are interdefinable (each is a first-order reduct of the other – or equivalently $\langle \mathbb{A} \rangle_{\mathsf{fo}} = \langle \mathbb{B} \rangle_{\mathsf{fo}}$).
(2) $\mathsf{Aut}(\mathbb{A}) = \mathsf{Aut}(\mathbb{B})$

2.1. **Primitivity and transitivity.**

Definition 3. Let $\mathcal{G} \subseteq \mathsf{Sym}(B)$ be a permutation group. A *congruence relation* of $\mathcal{G}$ is just an equivalence relation that is preserved by all permutations in $\mathcal{G}$. A *block* of a congruence is an equivalence class.

Example 4. Let B be any set and $\mathcal{G}$ any permutation group. We always have the following two congruences:

- The *trivial* congruence: The equivalence relation is equality, (the blocks have all size 1).
- The *universal* congruence: The equivalence relation here is universality, (there is only one block).

Proposition 5. *Let $\mathcal{G}$ be a permutation group on a set B and $S \subseteq B$. Then, the following are equivalent:*

(1) S is a block.
(2) For all $g \in \mathcal{G}$ either $g(S) = S$ or $g(S) \cap S = \emptyset$.

Proof.

Date: 23/10/23 – Notes written up by Aris Papadopoulos, following Section 4.2 of [**?**]. All mistakes are, of course, due to him.

(1) ⇒ (2) Suppose that $S \subseteq B$ is a block of the congruence C and $g(S) \cap S \neq \emptyset$. By assumption, there is some $t \in g(S) \cap S$. so there is some $s \in S$ such that $g(s) = t$. We have that:

$$\begin{aligned} r \in S \text{ if, and only if, } & (r, s) \in C \\ \text{if, and only if, } & (g(r), g(s)) = (g(r), t) \in C \\ \text{if, and only if, } & g(r) \in S, \end{aligned}$$

so $g(S) = S$.

(2) ⇒ (1) Define

$$C = \{(x, y) : x = y \text{ or } \exists g \in \mathcal{G} \text{ s.t. } g(x), g(y) \in S\}.$$

We claim that this is a congruence (and this suffices, since then clearly S is a block of C). We need only check transitivity. Say $(x, y), (y, z) \in C$ and without loss assume that each pair consists of distinct elements. Then there are $g_1, g_2 \in \mathcal{G}$ such that $g_1(x), g_1(y), g_2(y), g_2(z) \in S$. In particular:

$$g_2(y) \in (g_2 \circ g_1^{-1})(S),$$

and by (2), this means that $(g_2 \circ g_1^{-1})(S) = S$. Since $g_1(x) \in S$ we have that $(g_2 \circ g_1^{-1})(g_1(x)) = g_2(x) \in S$ and we are done.

□

Proposition 6. *If $\mathbb{B}$ is ω-cateogrical then the congruences of* $\mathsf{Aut}(\mathbb{B})$ *are precisely the f.o.-definable equivalence relations of $\mathbb{B}$.*

Definition 7. Let $\mathcal{G}$ be a permutation group on a set B. We say that $\mathcal{G}$ is:

(1) *primitive* if it has no congruences other than the trivial congruence and the universal congruence.
(2) *k-transitive* if for any two k-tuples of distinct elements $s, t \in B^k$ there is some $\alpha \in \mathcal{G}$ such that $\alpha s = t$.
(3) *transitive* if it is 1-transitive.
(4) *k-set transitive* if for any $S, T \subseteq B$ of cardinality k there is an $\alpha \in \mathcal{G}$ such that $\alpha(S) = T$.

2.2. **Group actions.**

Definition 8. A *(left) group action* of a group G on a set X is a function:

$$\cdot : G \times X \to X,$$

such that:

(1) $e \cdot x = x$, for all $x \in X$.
(2) $(gh) \cdot x = g \cdot (h \cdot x)$ for all $x \in X$ and all $g, h \in G$.

We denote group actions by $G \curvearrowright X$. We say that an action is *faithful* if for any two distinct $g, h \in G$ there is some $x \in X$ such that $g \cdot x \neq h \cdot x$.

Remark 9. Equivalently, a group action $G \curvearrowright X$ is just a group homomorphism $G \to \mathsf{Sym}(X)$. The action is faithful, if, and only if, that homomorphism is injective (i.e. $G \cong H$ for some $H \leq \mathsf{Sym}(B)$).

Examples. Let $G \curvearrowright X$. We can build two new actions:

(1) The *componentwise action* $G \curvearrowright X^n$ given by:

$$g \cdot (x_1, \dots, x_n) := (g \cdot x_1, \dots, g \cdot x_n),$$

for all $(x_1, \dots, x_n) \in X^n$.

(2) The *setwise action* $G \curvearrowright \binom{X}{n}$, given by:

$$g \cdot \{x_1, \dots, x_n\} := \{g \cdot x_1, \dots, g \cdot x_n\},$$

for all $\{x_1, \dots, x_n\} \in \binom{X}{n}$.

If $G \curvearrowright X$ faithfully, then both the actions above are also faithful.

Definition 10. Let $G \curvearrowright X$ and $x \in X$. The *orbit of x under G* is the set:

$$\{g \cdot x : g \in G\}.$$

Clearly, given a group action, we can consider its image in $\mathsf{Sym}(X)$ and think of it as a permutation group $\mathcal{G} \subseteq \mathsf{Sym}(B)$, and vice versa. All the terminology that was developed previously will now be used for abstract group actions.

2.3. **Products.** Let $G \curvearrowright X$ and $O \subseteq X$ be an orbit. Then we have a natural action $G_O \curvearrowright O$ given by restricting the permutations to the elements of O. This action is always transitive, but it is not necessarily faithful. We call the permutation group G_O a *transitive constituent* of G.

Proposition 11. *Let $G \curvearrowright X$ be a group action. Then G is a subcartesian product of its transitive constituents (that is, G is a subgroup of $\prod_{O \in \mathcal{O}} G_O$ and it projects onto each G_O).*

Definition 12. Let X_1, X_2 be disjoint sets and $G_i \curvearrowright X_i$. Then the action $G_1 \times G_2$ on $X_1 \sqcup X_2$ given by:

$$(g_1, g_2)z \mapsto \begin{cases} g_1 z \text{ if } z \in X_1 \\ g_2 z \text{ if } z \in X_2, \end{cases}$$

is called the *natural intransitive action* of $G_1 \times G_2$ on $X_1 \sqcup X_2$.

Proposition 13. *If $G_i \curvearrowright X_i$ oligomorphically for $i = 1, 2$ then the natural intransitive action of $G_1 \times G_2$ is also oligomorphic.*

Proof. Let $f_i(n)$ denote the number of orbits of the setwise action $G_i \curvearrowright X_i^n$. Clearly:

$$f(n) = \sum_{i=1}^{n} f_1(i) \times f_2(n-i).$$

Since both $f_1(n)$ and $f_2(n)$ are finite, for all $n \in \mathbb{N}$ it follows that so is $f(n)$. To show that $G_1 \times G_2$ is oligomorphic, we need to show that it has a finite number of orbits under the componentwise action, but this is immediate, since that number is bounded by $n! f(n)$. □

If G_1 and G_2 are the automorphism groups of ω-categorical relational structures $\mathbb{A}$ and $\mathbb{B}$ with disjoint domains A and B respectively then the natural intransitive action on $A \sqcup B$ can also be described as the automorphism group of a relational structure $\mathbb{C}$. If τ is the signature of $\mathbb{A}$ and σ is the signature of $\mathbb{B}$ then we can take for $\mathbb{C}$ the following:

- Signature: $\sigma \cup \tau \cup \{P\}$, where P is a new unary relation symbol.
- Domain: $A \sqcup B$.
- Interpretations: $R^{\mathbb{C}} = R^{\mathbb{A}}$ for $R \in \tau$, $R^{\mathbb{C}} = R^{\mathbb{B}}$ for $R \in \sigma$ and $P^{\mathbb{C}} = A$.

Remarks 14. Since reducts of ω-categorical structures are again ω-categorical, this shows, in particular, that the disjoint union of ω-categorical structures is ω-categorical.

Definition 15. Let $G_i \curvearrowright X_i$, for $i = 1, 2$. The *product action* $G_1 \times G_2 \curvearrowright X_1 \times X_2$ is given by:

$$(g_1, g_2)(x_1, x_2) := (g_1 x_1, g_2 x_2),$$

for all $g_i \in G_i$ and $x_i \in X_i$, for $i = 1, 2$.

Remark 16. If $G_i \curvearrowright X_i$ transitively, for $i = 1, 2$ then the product action is again transitive.

Proposition 17. *If $G_i \curvearrowright X_i$ oligomorphically for $i = 1, 2$ then the product action of $G_1 \times G_2$ is also oligomorphic.*

Proof. The number of orbits of $G_1 \times G_2$ (on n-tuples) is just the product of the number of orbits of G_1 and G_2 on n-tuples. □

Definition 18. Let $\mathbb{A}, \mathbb{B}$ be relational structures (in a posteriori) disjoint languages (both with a symbol for equality). We define their *algebraic product* $\mathbb{A} \boxtimes \mathbb{B}$ to be the structure on $A \times B$ containing the following relations:

$$\{((a_1, b_1), \dots (a_n, b_n)) : (a_1, \dots, a_n) \in R^{\mathbb{A}}, (b_1, \dots, b_n) \in B^n\},$$

for all n-ary R in the language of $\mathbb{A}$ and:

$$\{((a_1, b_1), \dots (a_n, b_n)) : (a_1, \dots, a_n) \in A^n, (b_1, \dots, b_n) \in R^{\mathbb{B}}\},$$

for all n-ary R in the language of $\mathbb{B}$.

Remark 19. In the notation above, we have relations for equality in both cooridinates (since each language has a symbol for equality). We will denote these by E_1 and E_2 (note that these are congruences of $\mathsf{Aut}(\mathbb{A} \boxtimes \mathbb{B})$.

Proposition 20. *Let $\mathbb{A}, \mathbb{B}$ be relational structures. Then:*

$$\mathsf{Aut}(\mathbb{A} \boxtimes \mathbb{B}) = \mathsf{Aut}(\mathbb{A}) \times \mathsf{Aut}(\mathbb{B}).$$

Moreover:

- *If $\mathbb{A}, \mathbb{B}$ are homogeneous, then so is $\mathbb{A} \boxtimes \mathbb{B}$.*
- *If $\mathbb{A}, \mathbb{B}$ are finitely bounded (recall this means that **the languages are finite** and their ages are of the form $\mathsf{Forb}^{emb}(\mathcal{F})$, for some finite set of finite structures $\mathcal{F}$) then so is $\mathbb{A} \boxtimes \mathbb{B}$.*

Proof.

- $\mathsf{Aut}(\mathbb{A} \boxtimes \mathbb{B}) = \mathsf{Aut}(\mathbb{A}) \times \mathsf{Aut}(\mathbb{B})$: On the one hand, it is clear that $\mathsf{Aut}(\mathbb{A}) \times \mathsf{Aut}(\mathbb{B}) \subseteq \mathsf{Aut}(\mathbb{A} \boxtimes \mathsf{Aut}(\mathbb{B})$, since any automorphism in the product of the two groups preserves all relations in the algebraic product of the two structures. Now for the converse, suppose that $g \in \mathsf{Aut}(\mathbb{A} \boxtimes \mathbb{B})$.
- *Homogeneity:* Fix a partial isomorphism i with domain $\{(a_1, b_1), \dots (a_n, b_n)\}$. Let i_1 be the restriction of i to the first coordinate and i_2 the restriction of i to the second. Thus i_1 is a partial isomorphism of $\mathbb{A}$ with domain $\{a_1, \dots, a_n\}$ and i_2 a partial isomorphism of $\mathbb{B}$ with domain $\{b_1, \dots, b_n\}$. By homogeneity, both of these extend to automorpshisms, say $\widehat{i_1}$ and $\widehat{i_2}$. By the previous item, the map $(\widehat{i_1}, \widehat{i_2}) \in \mathsf{Aut}(\mathbb{A} \boxtimes \mathbb{B})$, and we are done.
- *Finite boundedness:* To fix notation, say $\mathsf{Age}(\mathbb{A}) = \mathsf{Forb}^{emb}(\mathcal{A})$ and $\mathsf{Age}(\mathbb{B}) = \mathsf{Forb}^{emb}(\mathcal{B})$, where $\mathcal{A}, \mathcal{B}$ are finite sets of finite structures. Then:

  $$\mathsf{Age}(\mathbb{A} \boxtimes \mathbb{B}) = \mathsf{Forb}^{emb}(\mathcal{C}),$$

  where $\mathcal{C}$ consists of structures of the form $\hat{A} \boxtimes B_n$ and $A_n \boxtimes \hat{B}$, where $\hat{A} \in \mathcal{A}$ has size n and B_n ranges over all possible structures in the language of $\mathbb{B}$ of size n and $\hat{B} \in \mathcal{B}$ has size n and A_n ranges over all possible structures in the language of $\mathbb{B}$ of size n.

□

Remark 21. The algebraic product of structures is an associative operation, thus we can extend the definition to define the algebraic product of d-structures.

Remark 22. If $\mathbb{A}$ and $\mathbb{B}$ are structures with the same signature then $\mathsf{Aut}(\mathbb{A} \times \mathbb{B}) \supseteq \mathsf{Aut}(\mathbb{A} \boxtimes \mathbb{B})$, since any permutation of $A \times B$ which preserves all relations of $\mathbb{A} \boxtimes \mathbb{B}$ preserves all relations of $\mathbb{A} \times \mathbb{B}$. In particular, if the structures are ω-categorical, the fact that $\mathsf{Aut}(\mathbb{A} \boxtimes \mathbb{B})$ is oligomorphic implies that that $\mathsf{Aut}(\mathbb{A} \times \mathbb{B})$ is oligomorphic as well (orbits of the big group are partitioned into orbits of the smaller one) and thus $\mathbb{A} \times \mathbb{B}$ is ω-categorical too.

Upshot The class of all ω-categorical structures of some fixed signature considered up to homomorphic equivalence forms a lattice with the *homomorphism order* (the disjoint union is the join and the product is the meet).

References

[Bod21] Manuel Bodirsky. *Complexity of Infinite-Domain Constraint Satisfaction*. Cambridge University Press, May 2021.

Appendix A. Background Definitions

A.1. **sInv and Aut.** Let B be a countable set.

- If $\mathcal{P} \subseteq \mathsf{Sym}(B)$ then we define $\mathsf{sInv}(\mathcal{P})$ to be the set of all subsets of B^n (for $n \in \mathbb{N}$) which are preserved by all permutations in $\mathcal{P}$, and their inverses.
- If $\mathcal{R}$ is a collection of subsets of B^n (for $n \in \mathbb{N}$) we write $\mathsf{Aut}(\mathcal{R})$ for the subset of $\mathsf{Sym}(B)$ consisting of all permutations $g \in \mathsf{Sym}(B)$ such that both g and g^{-1} preserve R for all $R \in \mathcal{R}$.

If $\mathbb{A}$ is a relational structure we write $\langle \mathbb{A} \rangle_{\mathsf{fo}}$ for the set of all first-order definable subsets of A^n (for $n \in \mathbb{N}$).

A.2. **Products.** If $\mathbb{A}$ and $\mathbb{B}$ are relational structures with the same signature τ, we define their *direct* product to be the structure $\mathbb{A} \times \mathbb{B}$ which has domain $A \times B$ and:

$$R^{\mathbb{A} \times \mathbb{B}}((a_1, b_1), \ldots, (a_n, b_n)) \text{ if, and only if, } \begin{cases} R^{\mathbb{A}}(a_1, \ldots, a_n) \text{ and} \\ R^{\mathbb{B}}(b_1, \ldots, b_n), \end{cases}$$

for each n-ary relation symbol $R \in \tau$.