Countable categoricity & Fraïssé amalgamation

Recall $\mathbb{C}$ is homogeneous $\Leftrightarrow$ $\forall$ finite $\mathbb{A}, \mathbb{B} \subseteq \mathbb{C}$
$\forall f: \mathbb{A} \to \mathbb{B}$ -iso
$\exists \alpha \in Aut(\mathbb{C})$ s.t. $\alpha|_{\mathbb{A}} = f$

$\mathbb{C}$ is ω-categorical $\overset{R\text{-}N}{\Leftrightarrow}$ $\mathbb{C}$ is countably infinite and
$\forall n > 0$ $Aut(\mathbb{C}) \curvearrowright C^n$ has finitely many orbits

Lem1: $\mathbb{C}$ is (i) homogeneous, (ii) countably infinite, and
(iii) $\forall k > 0$ # k-ary relations defined by atomic formulas $< \infty$

Then, $\mathbb{C}$ is ω-categorical.

Pf: Homogeneity
$\Updownarrow$
[orbits of n-tuples $\xleftrightarrow{1\text{ to }1}$ isomorphism types of size n]

(iii) $\Leftrightarrow$ $\exists$ fin many isomorphism types of size n ($\forall n > 0$) $\square$

Homogeneity & Quantifier elimination

Def: $\mathbb{B}$ admits quantifier elimination if for any $\varphi(\bar{x}) \in FO$ there is $\psi(\bar{x})$ - quantifier-free s.t.

$$\forall \bar{a} \in B^n \left(\mathbb{B} \models \varphi(\bar{a}) \Leftrightarrow \mathbb{B} \models \psi(\bar{a}) \right)$$

Lem 2: $\mathbb{B}$ is ω-cat $\Rightarrow$ [$\mathbb{B}$ admits q.e. $\Leftrightarrow$ $\mathbb{B}$ is homogeneous]

Pf $\Longrightarrow$ Take $\bar{a}, \bar{b} \in B^k$ s.t. $\underbrace{\mathbb{B}[\bar{a}] \cong \mathbb{B}[\bar{b}]}_{*}$ (WTS $\bar{b} \in Orb(\bar{a})$)

By Ryll-Nardzewski, $\exists \varphi \in FO$ s.t. $\forall \bar{x} \in B^k$ $\mathbb{B} \models \varphi(\bar{x}) \Leftrightarrow \bar{x} \in Orb(\bar{a})$

By assumption, we can choose φ to be quantifier-free

$$\mathbb{B} \models \varphi(\bar{a}) \overset{*}{\Longrightarrow} \mathbb{B} \models \varphi(\bar{b}) \Rightarrow \bar{b} \in Orb(\bar{a})$$

$\Longleftarrow$ $\forall \varphi \in FO$ $\{\bar{b} \in B^k \mid \mathbb{B} \models \varphi(\bar{b})\} = Orb(\bar{a}_1) \cup \ldots \cup Orb(\bar{a}_n)$

WTS $\forall k > 0 \, \forall \bar{a} \in B^k \, \exists \psi_{\bar{a}} \in qf\,FO$ s.t. $\forall \bar{x}$ $\mathbb{B} \models \psi_{\bar{a}}(\bar{x}) \Leftrightarrow x \in Orb(\bar{a})$

$$\psi_{\bar{a}}(\bar{a}) := \bigwedge_{\substack{\bar{c} \subseteq \bar{a} \\ \mathbb{B} \models R(\bar{c})}} R(\bar{c}) \wedge \bigwedge_{\substack{\bar{d} \subseteq \bar{a} \\ \mathbb{B} \not\models R(\bar{d})}} \neg R(\bar{d})$$

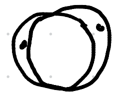
Clearly $\bar{b} \in Orb(\bar{a}) \overset{\text{assumpt.}}{\Longleftrightarrow} \mathbb{B} \models \psi_{\bar{a}}(\bar{b})$ $\square$

Strong amalgamation & Algebraic Closure

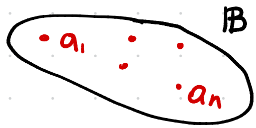
Recall

$$\begin{tikzcd} & \mathbb{C} & \\ \mathbb{B}_1 \arrow[ur, "g_1"] & & \mathbb{B}_2 \arrow[ul, "g_2"'] \\ & \mathbb{A} \arrow[ul, "f_1"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

strong if $g_1(B_1) \cap g_2(B_2) = g_1 \circ f_1(A)$

1-point if $|B_i \setminus A| = 1$ 

(strong) amalgamation $\Leftrightarrow$ (strong) 1-point amalgamation

$(\mathbb{B}, a_1 \ldots\ldots a_n)$ 

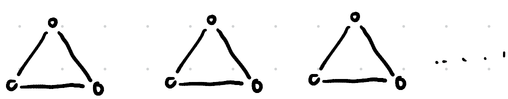
$A = \{a_1 \ldots\ldots a_n\}$

For $\varphi \in FO$, $X_\varphi := \{b \in B \mid (\mathbb{B}, a_1 \ldots\ldots a_n) \models \varphi(b)\}$

Def $acl_{\mathbb{B}}(A) := \bigcup_{\substack{\varphi \in FO \\ |X_\varphi| < \infty}} X_\varphi$ – algebraic closure of A in $\mathbb{B}$

$\mathbb{B}$ has no algebraicity if $\forall A \subset B$ s.t. $|A| < \infty$, $acl_{\mathbb{B}}(A) = A$

Example: $\mathbb{G} := \mathbb{K}_3 \cup \mathbb{K}_3 \cup \ldots\ldots$



- ω-cat
- homogeneous
- $Aut(\mathbb{G})$ has no finite orbits
- $\mathbb{G}$ has algebraicity

Prop: $\mathbb{B}$ has algebraicity $\Leftrightarrow \exists a_1 \ldots a_n \in B \; \exists b \in B \setminus \{a_1 \ldots a_n\}$ s.t. $\{b\}$ is definable in $(\mathbb{B}, a_1 \ldots a_n)$

Lem 3: $\left.\begin{array}{l}\mathbb{B} \text{ - (i) homogeneous} \\ \quad\text{ - (ii) } \omega\text{-cat}\end{array}\right\} \Rightarrow \left[\begin{array}{c}\mathbb{B} \text{ has no algebraicity} \\ \Updownarrow \\ \mathrm{Age}(\mathbb{B}) \text{ has strong amalgam.}\end{array}\right]$

$\mathrm{Age}(\mathbb{B}) = \{\mathbb{C} \mid |\mathbb{C}| < \infty,\ \mathbb{C} \subseteq \mathbb{B}\}$

Pf: $\Uparrow$ Suppose:

$\exists a_1 \dots a_n \in B\ \exists b \in B \setminus a_1 \dots a_n$ s.t.

$\{b\}$ is definable in $(\mathbb{B}, a_1 \dots a_n)$ by some φ

Let $\mathbb{A} := \mathbb{B}[a_1 \dots a_n]$ $\quad \mathbb{C} := \mathbb{B}[a_1 \dots a_n b]$

$$\begin{tikzcd} & \mathbb{D} & \\ \mathbb{C} \arrow[ur, "g_1"] & & \mathbb{C} \arrow[ul, "g_2"'] \\ & \mathbb{A} \arrow[ul, "id"] \arrow[ur, "id"'] & \end{tikzcd}$$

- strong $\Rightarrow g_1(b) \neq g_2(b) \Rightarrow \alpha_1(b) \neq \alpha_2(b)$

$(i) \Rightarrow \exists \alpha_1 \alpha_2 \in \mathrm{Aut}(\mathbb{B})$ s.t. $\alpha_i|_{\mathbb{C}} = g_i \Rightarrow \alpha_1(a_j) = \alpha_2(a_j)$

$\alpha_2^{-1}\alpha_1(b) \neq b$ but $\alpha_2^{-1} \circ \alpha_1 \in \mathrm{Aut}(\mathbb{B}, a_1 \dots a_n)$

$\Downarrow$

$(\mathbb{B}\ a_1 \dots a_n) \models \varphi(\alpha_2^{-1}\alpha_1(b))$ $\lightning$

$\Downarrow$ Take $\mathbb{A}, \mathbb{B}_1, \mathbb{B}_2 \in \mathrm{Age}(\mathbb{B})$ s.t. $B_i \setminus A = \{b_i\}$

$(i) \Rightarrow$
$$\begin{tikzcd} & \mathbb{C} & \\ \mathbb{B}_1 \arrow[ur, "e_1"] & & \mathbb{B}_2 \arrow[ul, "e_2"'] \\ & \mathbb{A} \arrow[ul, "id"] \arrow[ur, "id"'] & \end{tikzcd}$$

WTS $\mathbb{C}$ strong

Let $A = \{a_1 \dots a_n\}$

$(ii) \Rightarrow \mathrm{Orb}(e_1(b_1))$ wrt $\mathrm{Aut}(\mathbb{B}\ a_1 \dots a_n)$ is FO-definable $\Longrightarrow$

$\underset{\text{no alg}}{\Longrightarrow} |\mathrm{Orb}(e_1(b_1))| > 1 \Rightarrow$ we can choose $e_2(b_2)$ s.t.

$\big(e_2(b_2) \in \mathrm{Orb}(e_1(b_1))$ & $e_2(b_2) \neq e_1(b_1)\big) \Rightarrow$ strong $\square$

Application of "no algebraicity"

Lem 4: Let $\mathbb{B}$ be (I) ω-cat (II) no algebraicity, and (III) $\forall R \in \tau\ \forall \bar{a} \in R^{\mathbb{B}}\ \forall a_i, a_j \in \bar{a}\quad a_i \neq a_j$ ($\bar{a}$ is injective)

Then, $\forall \mathbb{A}\ |A| < \infty\ \left[\mathbb{A} \to \mathbb{B} \iff \mathbb{A} \xrightarrow{\text{injective}} \mathbb{B}\right]$

Pf: follows from Lem 3 $\square$

HUBIČKA & NEŠETŘIL (2010s)

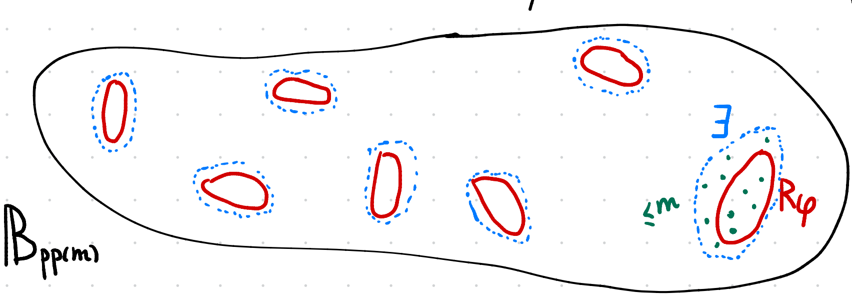
$\mathcal{F}$ = finite set of finite connected τ-structures

$m := \max\limits_{\mathbb{F} \in \mathcal{F}} |F|$

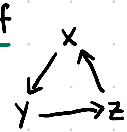
$\text{Forb}_{\text{hom}}(\mathcal{F}) := \{\mathbb{A} \mid |A| < \infty\ \&\ \forall \mathbb{F} \in \mathcal{F}\ \ \mathbb{F} \not\to \mathbb{A}\}$

$PP(m)$:= all pp-formulas $\varphi(\bar{x})$ s.t.
(I) φ is connected
(II) φ has $\leq m$ vars
(III) $|\bar{x}| \geq 1$

For $\mathbb{B}$ – τ-structure, $\mathbb{B}_{pp(m)}$:= expansion by all relations definable by $\varphi \in PP(m)$.



Def



ccq(·) ↓ ↑ cd(·)

$\exists xyz\ Exy \wedge Eyz \wedge Ezx$

Def: $\mathcal{P}$:= set of all substructures of $\mathbb{A}_{pp(m)}$,
where $\mathbb{A} \in Forb_{hom}(\mathcal{F})$.

Lem 5: $\mathcal{P}$ has strong amalgamation.

Construction: (I) Take $\mathbb{A}, \mathbb{B}_1, \mathbb{B}_2 \in \mathcal{P}$ s.t. $\mathbb{A} \subseteq \mathbb{B}_1$ & $\mathbb{A} \subseteq \mathbb{B}_2$

(II) $\varphi_1^0 := ccq(\mathbb{B}_1)$ $\quad \varphi_2^0 := ccq(\mathbb{B}_2)$

PP(m)

(III) If $R_\psi(\bar{x})$ — conjunct of φ_i^0, then replace it with $\psi(\bar{x})$

e.g. $\psi(x) = \exists y\, Exy$ $\quad \varphi_i^0 = \exists xy\, Exy \wedge R_\psi(y) \longrightarrow \exists xyz\, Exy \wedge Eyz$

(IV) Denote the resulting sentences by φ_1 φ_2

(V) $\mathbb{C} := cd(\varphi_1 \wedge \varphi_2)$ $\quad \mathbb{C}_{pp(m)}$ — required strong amalgam

$$
\begin{tikzcd}
 & \mathbb{C}_{pp(m)} & \\
\mathbb{B}_1 \arrow[ur] & & \mathbb{B}_2 \arrow[ul] \\
 & \mathbb{A} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Claim 1: $\mathbb{C} \in Forb_{hom}(\mathcal{F})$

Claim 2: $\mathbb{B}_1 \xhookrightarrow{id} \mathbb{C}_{pp(m)}$ $\quad \mathbb{B}_2 \xhookrightarrow{id} \mathbb{C}_{pp(m)}$

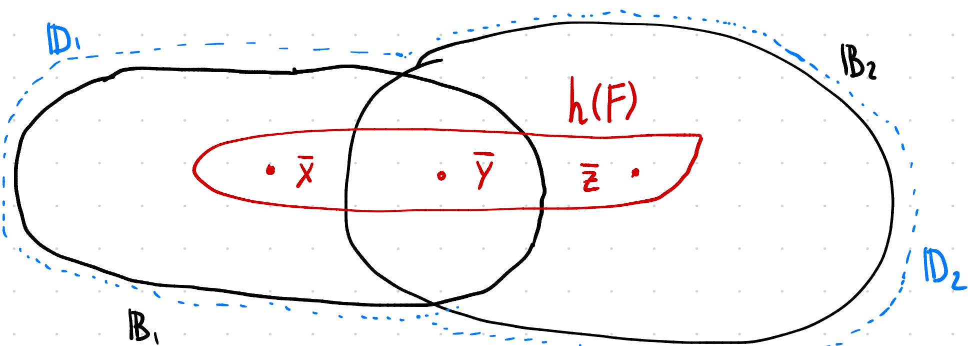
Pf(Claim 1): • Suppose $\exists \mathbb{F} \in \mathcal{F}$ s.t. $\mathbb{F} \xrightarrow{h} \mathbb{C}$

• Denote $\mathbb{D}_1, \mathbb{D}_2 \in \mathrm{Forb}_{hom}(\mathcal{F})$ s.t. $\mathbb{B}_i \subseteq \mathbb{D}_{i\,pp(m)}$ $\quad \mathbb{D}_i := cd(\varphi_i)$

• If $h(F) \subseteq B_i$, then $h(F) \subseteq D_i$, then $\mathbb{F} \to \mathbb{D}_i$ ↯

• So $h(F) \cap (B_i \setminus A) \neq \emptyset$

• $\mathbb{F}$-connected $\Rightarrow h(F) \cap A \neq \emptyset$ $\quad ccq(\mathbb{F}) := \exists \bar{x}\bar{y}\bar{z}\, \theta_1(\bar{x}\bar{y}) \wedge \theta_2(\bar{y}\bar{z})$



• $|F| \leq m \Rightarrow \psi_1(\bar{y}) := \exists \bar{x}\, \theta_1(\bar{x}\bar{y}) \in PP(m)$ & $\psi_2(\bar{y}) := \exists \bar{z}\, \theta_2(\bar{y}\bar{z}) \in PP(m)$

• Denote $\mathbb{A}' \in \mathrm{Forb}_{hom}(\mathcal{F})$ s.t. $\mathbb{A} \subseteq \mathbb{A}'_{pp(m)}$

• $\mathbb{B}_i \models \psi_i(\bar{y}) \Rightarrow \mathbb{B}_i \models R_{\psi_i}(\bar{y}) \Rightarrow \mathbb{A} \models R_{\psi_i}(\bar{y}) \Rightarrow \mathbb{A}' \models ccq(\mathbb{F})$

↯

Made with Goodnotes

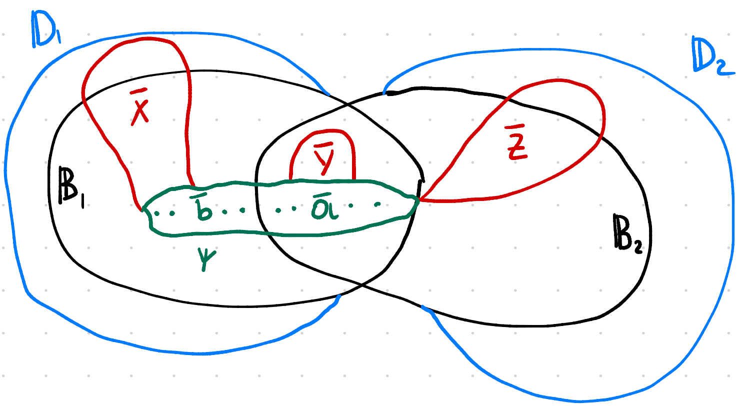
Pf(Claim 2): WTS $\mathbb{B}_1, \mathbb{B}_2 \subseteq \mathbb{C}_{pp(m)}$

- $\mathbb{B}_1 \models R_\psi(\bar{a}) \Rightarrow \mathbb{D}_1 \models \psi(\bar{a}) \Rightarrow \mathbb{C} \models \psi(\bar{a}) \Leftrightarrow \mathbb{C}_{pp(m)} \models R_\psi(\bar{a})$

- Now suppose, for some $\begin{matrix} \bar{a} \subseteq A \\ \bar{b} \subseteq B_1 \setminus A \end{matrix}$ $\mathbb{C} \models \psi(\bar{a}\bar{b})$

- WLOG $\psi(\bar{a}\bar{b}) = \exists \bar{x}\bar{y}\bar{z}\left(\psi_1(\bar{a}\bar{b}\bar{x}\bar{y}) \wedge \psi_2(\bar{a}\bar{y}\bar{z})\right)$, where

(i) $\bar{x} \subseteq Vars(\varphi_1) \setminus Vars(\varphi_2)$ (ii) $\bar{y} \subseteq V(\varphi_1) \cap V(\varphi_2)$ (iii) $\bar{z} \subseteq V(\varphi_2) \setminus V(\varphi_1)$



- $\theta(\bar{a}\bar{y}) := \exists \bar{z}\, \psi_2(\bar{a}\bar{y}\bar{z})$ has $\leq m$ vars

- $\mathbb{D}_2 \models \theta(\bar{a}\bar{y}) \Rightarrow \mathbb{B}_2 \models R_\theta(\bar{a}\bar{y}) \Rightarrow \mathbb{A} \models R_\theta(\bar{a}\bar{y}) \Rightarrow \mathbb{B}_1 \models R_\theta(\bar{a}\bar{y}) \Rightarrow$

$\Rightarrow \mathbb{D}_1 \models \theta(\bar{a}\bar{y}) \Rightarrow \mathbb{D}_1 \models \psi(\bar{a}\bar{b}) \Rightarrow \mathbb{B}_1 \models R_\psi(\bar{a}\bar{b})$

$\square$

Main result of HN

Th: There is $\mathbb{B}$ – ω-cat, without algebraicity s.t.

(I) $Age(\mathbb{B}) = Forb_{hom}(\mathcal{F})$

(II) $\mathbb{B}_{pp(m)}$ – homogeneous

(III) $Age(\mathbb{B}_{pp(m)}) = \mathcal{P}$

Pf: Let $\mathbb{B}' :=$ Fraïssé-limit$(\mathcal{P})$ & $\mathbb{B} := \tau$-reduct of $\mathbb{B}'$

WTS $\mathbb{B}_{pp(m)} = \mathbb{B}'$

- take $\psi(x_1 \dots x_n) \in PP(m)$.
- By def, for all $a_1 \dots a_n \in B$, $\mathbb{B} \models \psi(a_1 \dots a_n) \Leftrightarrow \mathbb{B}_{pp(m)} \models R_\psi(a_1 \dots a_n)$
- Suffices to show: $\mathbb{B}' \models R_\psi(a_1 \dots a_n) \Leftrightarrow \mathbb{B} \models \psi(a_1 \dots a_n)$
- $\mathbb{A}' := \mathbb{B}'[a_1 \dots a_n] \Rightarrow \mathbb{A} \in \mathcal{P} \Rightarrow \mathbb{A}' \subseteq \mathbb{A}_{pp(m)}$ s.t. $\mathbb{A} \in Forb_{hom}(\mathcal{F})$
- $\mathbb{B}' \models R_\psi(a_1 \dots a_n) \Leftrightarrow \mathbb{A}' \models R_\psi(a_1 \dots a_n) \Leftrightarrow \mathbb{A} \models \psi(a_1 \dots a_n)$
- $\mathbb{A}_{pp(m)} \in \mathcal{P} \Rightarrow \mathbb{A}_{pp(m)} \hookrightarrow \mathbb{B}' \xRightarrow{\mathbb{B}'\text{-homog.}} \mathbb{B} \models \psi(a_1 \dots a_n)$ $\square$

Cherlin-Shelah-Shi (1999)

Th: There exists a countable model-complete $\mathbb{B}$ s.t.

(I) $\mathrm{Age}^*(\mathbb{B}) = \mathrm{Forb}^*_{hom}(\mathcal{F})$

$*$ contain all countable strs. not only finite

(II) $\mathbb{B}$ - ω-cat, has no algebraicity