SOME REMINDERS for TODAY

$\forall^-$-sentence $\equiv \forall x(\varphi(x) \to \bot)$
φ qf

pe-formula (or $\exists^+$-formula) $\equiv \exists \bar{x}\, \varphi(\bar{x}, y)$
φ qf with no $\neg$

BASIC FACTS ABOUT $T_{\forall^-}$

- For φ $\exists^+$-sentence, $T \cup \{\varphi\}$ is sat iff $T_{\forall^-} \cup \{\varphi\}$ is sat.
- $T_{\forall^-} = S_{\forall^-}$ iff every model of T maps homom to a model of S & vice versa

CONTINUATION TO ep-closed models

For $\kappa \geq \max(|T|, \aleph_0)$, $A \models T$, $|A| \leq \kappa$, there is a T-ep-closed B s.t. $|B| \leq \kappa$ and A maps homomorphically to B.

B is T-ep-closed iff for every $n \in \mathbb{N}$ every ep-n-type in A is a maximal ep-type of T.

SOME REMINDERS (cont)

EQUIVALENTS TO BEING A MC-CORE Let $\mathbb{B}$ be ctable ω-categorical. Then, tfae:

① $\mathbb{B}$ is a model complete core

All endomorphisms of $\mathbb{B}$ are embeddings

③ Every f.o.-formula is equivalent to an $\exists^+$-formula

⑤ $\overline{\mathrm{Aut}(\mathbb{B})} = \mathrm{End}(\mathbb{B})$

For any $e \in \mathrm{End}(\mathbb{B})$ and $t \in \mathbb{B}^n$ there is $\alpha \in \mathrm{Aut}(\mathbb{B})$ s.t. $\alpha t = e(t)$.

⑧ $\mathbb{B}$ has a model complete core theory.

All homomorphisms between models of T are elementary embeddings

§ 4.6 EXISTENTIALLY-POSITIVE RYLL-NARDZEWSKI

Let T be a satisfiable theory

Let ϕ_1, ϕ_2 be e.p. formulas with n free variables.

$\phi_1 \sim_n^T \phi_2$ iff for all ep-formulas ψ in free $x_1 \dots x_n$
$\{\phi_1, \psi\} \cup T$ is SAT iff $\{\phi_2, \psi\} \cup T$ is SAT.

The INDEX of an equivalence relation is the # of its classes

BASIC FACTS ABOUT $\sim_n^T$

ⓐ $U_{\forall^-} = T_{\forall^-} \Rightarrow (\phi_1 \sim_n^U \phi_2$ iff $\phi_1 \sim_n^T \phi_2)$

ⓑ Every maximal ep-n-type p is determined entirely by the $\sim_n^T$-classes of ep-n-formulas in p.

Proof:

ⓐ $\{\phi_1, \psi\} \cup T$ is sat iff $\{\phi_1, \psi\} \cup T_{\forall^-}$ is sat iff $\cdots$ iff $\{\phi_1, \psi\} \cup U$ is sat.

ⓑ p, q max. $\forall \phi_1 \in p \;\exists \phi_1' \in q$ s.t. $\phi_1 \sim_n^T \phi_1'$ and vice versa.

$S \subseteq q$ finite $\phi_1 \cup S \cup T$ is sat iff $\phi_1' \cup S \cup T$ is sat

By compactness $\phi_1 \cup q \cup T$ is sat $\overset{\text{max}}{\Rightarrow} \phi_1 \in q$. so $p = q$ $\boxtimes$

EP-RYLL NARDZEWSKI (CONSTRUCTION of the CORE)

Let T be satisfiable, in a countable rel signature and with JHP. tfae

1. T has an ω-cat mc. core companion $\Leftrightarrow \exists \mathbb{B}$ s.t. for all $\exists^+ \phi$ $T \cup \{\phi\}$ is sat iff $B \models \varphi(\bar{a})$ for some $\bar{a}$.
2. $\sim_n^T$ has finite index for each n.
3. T has fin many maximal ep-types in each n.
4. There is a (finite or countable) ω-categorical model complete core $\mathbb{B}$ s.t. for $\exists^+ \varphi$, $T \cup \{\varphi\}$ is sat iff $B \models \varphi$.

proof:

$1 \Rightarrow 2$ U be the mc-core comp of T. $U_{\forall^-} = T_{\forall^-}$.
So $\varphi_1 \sim_n^T \varphi_2$ iff $\varphi_1 \sim_n^U \varphi_2$ by (a).
$\varphi_1 \equiv_n^T \varphi_2 \Rightarrow \varphi_1 \sim_n^T \varphi_2$. If $\sim_n^T$ has infinite index, then there would be inf many $\equiv_n^U$-inequivalent formulas in U $\cancel{\times}$ ω-categoricity of U. ∎

$2 \Rightarrow 3$: By (b) ✓

$4 \Rightarrow 1$: $\mathbb{B}$ is a mc core iff $\mathbb{B}$ has a mc core theory

NTS: $Th(\mathbb{B})_{\forall^-} = T_{\forall^-}$

$T \models \forall x(\varphi(x) \to \bot)$ iff $T \cup \{\varphi\}$ is uns. iff $B \not\models \varphi(\bar{a})$ for any $\bar{a}$ iff $Th(\mathbb{B}) \models \forall x(\varphi(x) \to \bot)$ ✓

③ ⇒ ④: By JHP there is $\mathbb{C}$ (ctble or finite) s.t. for $\exists^+$ ϕ
$T \cup \{\phi\}$ is sat iff $\mathbb{C} \models \phi(\bar{a})$ for some $\bar{a} \in \mathbb{C}$. $Th(\mathbb{C})_{\forall^-} = T_{\forall^-}$

- $\mathbb{C}$ is finite $\rightsquigarrow$ take the core of $\mathbb{C}$. $\mathbb{B}$ has all of the required properties.
- $\mathbb{C}$ is ctble. take a hom of $\mathbb{C}$ into pe-closed model $\mathbb{B}$ also ctble.
  $Th(\mathbb{B})_{\forall^-} = T_{\forall^-}$

CLAIM: $\text{ep-tp}(\bar{a}) = \text{ep}(\bar{a}')$. Then, there is $f \in Aut(\mathbb{B})$ s.t. $f(\bar{a}) = \bar{a}'$.

Proof: $b \in B \setminus \bar{a}$. $p := \text{ep tp}(\bar{a}\, b)$ & p is maximal (being in ep model)

For each q ep-max $n+1$-type $\phi_q(\bar{x}, y)$ $\exists^+$ in p and not in q.
Since there are fin many such type take conjunction $\equiv$ $\exists^+$-form $\phi(\bar{x}, y)$
$\phi(\bar{x}, y) \in p$ by maximality.
$\exists y\, \phi(\bar{x}, y) \in \text{ep tp}(\bar{a})$ so $\in \text{ep tp}(\bar{a}')$.
there is b' s.t. $\mathbb{B} \models \phi(\bar{a}'\, b')$. Let $f(b) = b'$.
By construction $\text{ep tp}(\bar{a}\, b) = \text{ep tp}(\bar{a}'\, b')$.

Finish claim: $\mathrm{eptp}(\bar{a}) = \mathrm{eptp}(\bar{a}') \Rightarrow \bar{a} \equiv \bar{a}'$.

Finitely many max ep n-types $\to$ these are dist. by formulas

Since ep types determine types, types are isolated by ep-form.

- $\mathbb{B}$ is ω-cat
- $\mathbb{B}$ is a mc core

every fo formula is eq to an ep-one

$\blacksquare$

Note: $(\mathbb{Z}, <)$ has mc core$\overset{\omega\text{-cat}}{\checkmark}$ $(\mathbb{Q}, <)$.

inf many 2-types
but fin many
ep n-types for each n.

Lemma Let $\mathbb{B}$ and $\mathbb{C}$ be countable ω-categorical with $Th(\mathbb{B})_{\forall -} = Th(\mathbb{C})_{\forall -}$. Then, $\mathbb{B}$ and $\mathbb{C}$ are homomorphically eq.

proof: $\mathbb{C} \xrightarrow{hom} \mathbb{B}$.

Lemma 4.1.7 $\mathbb{C} \xrightarrow{hom} \mathbb{B}$ iff all fin. substructures of $\mathbb{C}$ map hom to $\mathbb{B}$.

$\bar{c} \in \mathbb{C}$ $\quad p = eptp(\bar{c})$

$\mathbb{B} \models p(\bar{b})$ for some tuple by ω-sat. $\bar{c}$ map hom to $\mathbb{B}$ $\bar{c} \to \bar{b}$.

$\blacksquare$

THEOREM 4.7.4 Every countable ω-categorical structure $\mathbb{B}$ is homomorphically equivalent to an ω-categorical model complete core $\mathbb{C}$. This is unique up to isomorphism.

proof: $Th(\mathbb{B})$ meets the req of ep-Ryll-Nordz. So there is mc core companion S which is ω-cat.

$\blacksquare$

Note: the model-complete core $\mathbb{C}$ of $\mathbb{B}$ embeds into $\mathbb{B}$.

Let $\mathbb{B} \xrightarrow{h} \mathbb{C}$ $\quad \mathbb{C} \xrightarrow{i} \mathbb{B}$. $h \circ i$ is an embedding $\Rightarrow i$ is an embedding.

TRANSFER OF PROPERTIES

Let $\mathbb{B}$ be stable ω-categorical and $\mathbb{C}$ be its mc-core.

1. $\mathbb{B}$ is homogeneous $\Rightarrow$ $\mathbb{C}$ is homogeneous

2. Let $i : \mathbb{C} \to \mathbb{B}$ be a homomorphism. $t_1, t_2 \in \mathbb{C}^n$ are s.t. $t_1 \equiv t_2$. Then $\exists\, e_1, e_2 \in \mathrm{End}(\mathbb{B})$ s.t.
$$e_1(i(t_1)) = i(t_2) \quad \text{and} \quad e_2(i(t_2)) = t_1$$

3. $i(t_1) \equiv i(t_2) \Rightarrow t_1 \equiv t_2$

4. For every n,
\# orbits of n-tuples under $\mathrm{Aut}(\mathbb{C}) \leq$ \# orbits of n-tuples under $\mathrm{Aut}(\mathbb{B})$.

5. If we have equality, $\mathbb{B} \cong \mathbb{C}$.

Proof: 3 $\Rightarrow$ 4.   3 $\Rightarrow$ 1

negations of f.o. formulas are $\equiv$ $\exists^+$-fmls in $\mathbb{C}$

$$qftp(t_1) = qftp(t_2) \xRightarrow{Th(\mathbb{C})_\forall = Th(\mathbb{B})_\forall} qftp(i(t_1)) = qftp(i(t_2))$$

$$\xRightarrow{\mathbb{B}\text{ hom}} i(t_1) \equiv i(t_2) \xRightarrow{3} t_1 \equiv t_2 .$$

② $i: \mathbb{C} \longrightarrow \mathbb{B}$ a homomorphism. $t_1, t_2 \in \mathbb{C}^n$ $\quad t_1 \equiv t_2$.

Then, $\exists e_1, e_2 \in \mathrm{End}(\mathbb{B})$ s.t.

$$e_1(i(t_1)) = i(t_2) \quad \text{and} \quad e_2(i(t_2)) = t_1$$

③ $i(t_1) \equiv i(t_2) \Rightarrow t_1 \equiv t_2$.

φ_1 and φ_2 be pp-defs of the orbits of t_1 and t_2 ($\mathbb{C}$ is a core)

$$i(t_1) \equiv i(t_2) \Longrightarrow i(t_2) \models \varphi_1 \Longrightarrow \varphi_1 \equiv \varphi_2 \Rightarrow t_1 \equiv t_2.$$

pp-form are preserved by homom.

If they are diff orbits

$$\mathbb{C} \models \forall \bar{x} \, (\varphi_1(\bar{x}) \wedge \varphi_2(\bar{x})) \longrightarrow \bot)$$

But $\mathbb{B}$ would also satisfy this formula.

(5) If $\forall n$ # orbits of n-tuples under $\mathrm{Aut}(\mathbb{C})$ = # orbits of n-tuples under $\mathrm{Aut}(\mathbb{B})$ then $\mathbb{B} \cong \mathbb{C}$.

<u>Proof</u>: We prove that $\mathbb{B}$ is a mc core and so $\cong \mathbb{C}$ by showing $\overline{\mathrm{Aut}(\mathbb{B})} = \mathrm{End}(\mathbb{B})$.

For each $\mathrm{Aut}(\mathbb{C})$-orbit O let s_O be a representative.

$$I : \{\text{orbits of } n\text{-tuples in } \mathbb{C}\} \to \{\text{orbits of } n\text{-tuples in } \mathbb{B}\} \qquad O \longmapsto \text{orbit of } i(s_O)$$

By (3) I is an injection, it is also a bijection by finiteness & same # of orbits.

Let $t \in \mathbb{B}^n$, $e \in \mathrm{End}(\mathbb{B})$. Choose
$s \in \mathbb{C}$ from preimage of the orbit of t
$s' \in \mathbb{C}$ // orbit of $e(t)$

$\exists\, \alpha, \beta \in \mathrm{Aut}(\mathbb{B})$ s.t. $\alpha\, i(s) = t \quad \beta\, i(s') = e(t)$.

Since $\mathbb{C}$ is a mc core

$\underbrace{h}_{\mathbb{B} \to \mathbb{C}} \circ\, e \circ \alpha \circ \underbrace{i}_{\mathbb{C} \to \mathbb{B}} \in \overline{\mathrm{Aut}(\mathbb{C})}$ so s and $h \circ e \circ \alpha \circ i(s)$ are in same $\mathrm{Aut}(\mathbb{C})$-orbit.

$h \circ e \circ \alpha \circ i(s) = h \circ e(t) = \underbrace{h \circ \beta \circ i}_{\in \mathrm{End}(\mathbb{C})}(s')$ so s and s' are in same $\mathrm{Aut}(\mathbb{C})$-orbit.

But then, by choice of s and s' (and since I is a map between orbits) $e(t)$ is in the same orbit as t. So $\overline{\mathrm{Aut}(\mathbb{B})} = \mathrm{End}(\mathbb{B})$ ∎