Chapters 5.1. and 5.2.

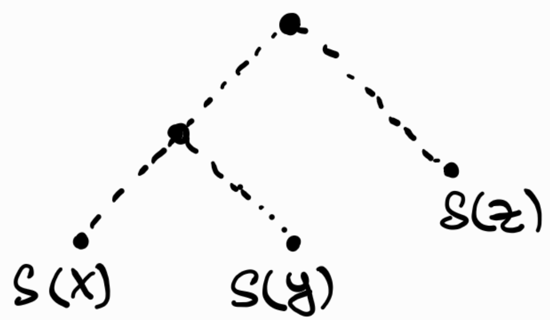
Chapter 5: interesting examples of ω-cat. CSPs that are <u>not</u> covered later in detail

5.1. Phylogeny Constraints and homogeneous C-relations

Rooted-Triple Satisfiability

INPUT: variables V, triples $xy|z$ for some $x,y,z \in V$
QUESTION: $\exists$ rooted tree T and map $s: V \to L(T)$ s.t.

$$\forall xy|z: \; \mathrm{yca}(s(x),s(y)) \text{ lies below } \mathrm{yca}(s(x),s(z))$$

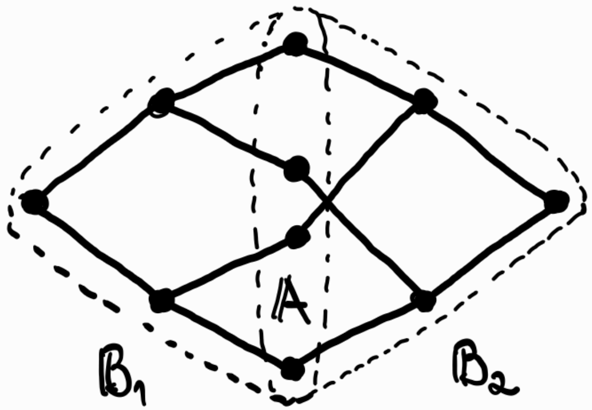


5.1.1. Leaf structures

T binary rooted tree $:\Leftrightarrow$ connected acyclic undirected gr.,
one vertex has degree $\in \{0,2\}$,
other vertices have degree $\in \{1,3\}$

$\mathcal{T} :=$ class of all finite binary rooted trees

- $\mathcal{T}$ not closed under substructures :(
- closure of $\mathcal{T}$ under substructures not an amalgamation class >:(



Def 5.1.1 • **Leaf structure** of $T \in \mathcal{T}$: $\mathbb{L}(T) := (L(T); |)$

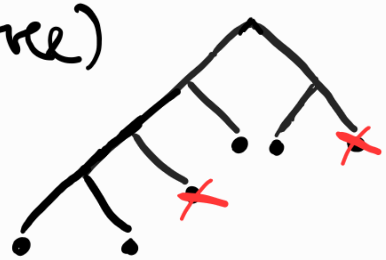
$ab|c :\Longleftrightarrow \mathrm{yca}(a,b)$ lies below $\mathrm{yca}(a,b,c)$ in T

- $\mathcal{C} := \{ \mathbb{L}(T) \mid T \in \mathcal{T} \}$
- for $\mathbb{L} \in \mathcal{C}$: $T(\mathbb{L})$:= the **underlying tree** of $\mathbb{L}$

Prop. 5.1.2 $\mathcal{C}$ is an amalgamation class.

Proof. • closure under isomorphisms ✓ (trivial)

- closure under substructures ✓ (ignoring some leaves leaves a tree)



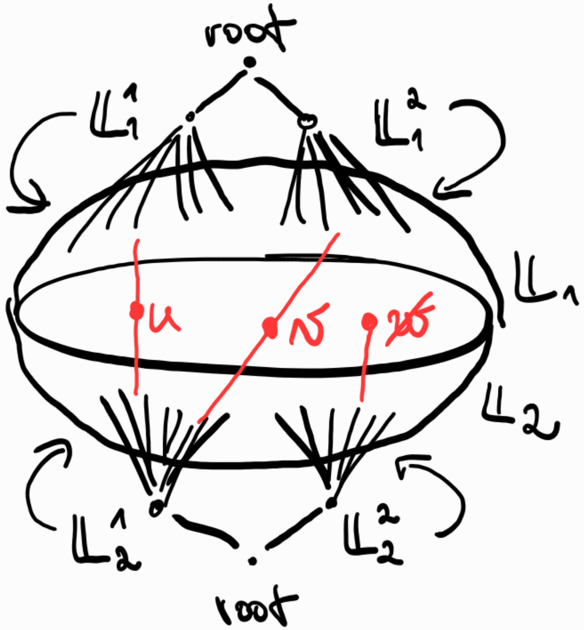
- for the AP, let $(\mathbb{L}_1, \mathbb{L}_2)$ be an amalgamation diagram for $\mathcal{C}$

- assume that the statement holds for all amalgamation diagrams (L_1', L_2') with $L_1' \cup L_2' \subsetneq L_1 \cup L_2$

- for $i \in \{1,2\}$:

  - $\mathbb{L}_i^1 :=$ the substructure of $\mathbb{L}_i$ left from the root in $\Pi(\mathbb{L}_i)$
  - $\mathbb{L}_i^2 :=$ ——— " ——— right ——— " ———

<u>Case 1</u>: $\exists u \in L_1^1 \cap L_2^1$ and $\exists v \in L_2^1 \cap L_1^2$



$\not\exists w \in L_2^2 \cap L_1$, because

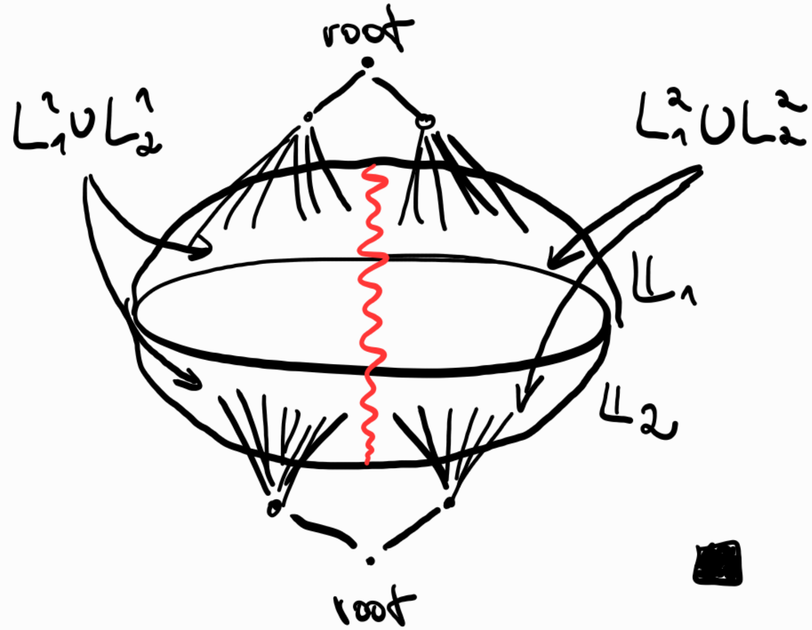
i) $\mathbb{L}_2 \models uv | w$
ii) $w \in L_1^1 \Rightarrow \mathbb{L}_1 \models uw | v$
iii) $w \in L_1^2 \Rightarrow \mathbb{L}_1 \models vw | u$

$\Rightarrow$ ↯

- $\mathbb{C} :=$ amalgam of $(\mathbb{L}_1, \mathbb{L}_2^1)$ ($\exists$ by assumption &)
- $\Pi :=$ the tree with $\Pi(\mathbb{C})$ left from the root,
  $\Pi(\mathbb{L}_2^2)$ right —— " ——
- clearly $\mathbb{L}(\Pi) \in \mathcal{C}$ is an amalgam for $(\mathbb{L}_1, \mathbb{L}_2)$ (by)

<u>Case 2</u>: $(L_1^1 \cup L_2^1) \cap (L_1^2 \cup L_2^2) = \emptyset$

- Similarly to case 1, join amalgams for $(\mathbb{L}_1^1, \mathbb{L}_2^1)$ and $(\mathbb{L}_1^2, \mathbb{L}_2^2)$, which exist by the assumption



∎

- $\mathbb{L} :=$ the Fraïssé limit of $\mathcal{C}$

$\mathbb{L}$ m.c. core because it has quantifier-elim. and:

$$\neg(x=y) \iff (xx|y),$$
$$\neg(xy|z) \iff (yz|x) \vee (xz|y) \vee (x=y=z)$$

CSP($\mathbb{L}$) is the Rooted-Triple Satisfiability problem

branching degree irrelevant

5.1.2 C-relations

Def A relation $C \subseteq L^3$ is a C-relation if $\forall a,b,c,d \in L$:

$\boxed{C_1}$ $C(a;b,c) \Rightarrow C(a;c,b)$

$\boxed{C_2}$ $C(a;b,c) \Rightarrow \neg C(b;a,c)$

$\boxed{C_3}$ $C(a;b,c) \Rightarrow C(a;d,c) \vee C(d;b,c)$

$\boxed{C_4}$ $a \neq b \Rightarrow C(a;b,b)$

- C is binary branching if

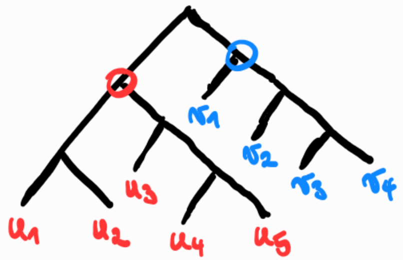
$$\forall x,y,z \left(x \neq y \vee x \neq z \vee y \neq z \Rightarrow \left(C(x;y,z) \vee C(y;x,z) \vee C(z;x,y) \right) \right)$$

Age($\mathbb{L}$) axiomatized by $\boxed{C1 - C4}$ + binary branching

$\Rightarrow \mathbb{L}$ is finitely bounded

<u>Remark</u> Th($\mathbb{L}$) a model companion of Th(binary branching C-relations)

5.1.3 The quartet satisfiability problem



<u>Def. 5.1.4.</u> If $\Pi \in \mathcal{T}$, $u_1, \dots, u_k, v_1, \dots, v_\ell \in L(\Pi)$, then

$\{u_1, \dots, u_k\} | \{v_1, \dots, v_\ell\} :\Leftrightarrow$ yca($\{u_1, \dots, u_k\}$) and yca($\{v_1, \dots, v_\ell\}$) unrelated in Π

<u>Def.</u> $Q :=$ the 4-ary relation defined in $\mathbb{L}$ by

$$(xy|uv) \vee (uv|x \wedge vx|y) \vee (xy|u \wedge yu|v)$$

CSP$(\mathbb{L};Q)$ is the following problem:

<u>Quartet Satisfiability</u>

INPUT: variables V, quadruples $xy:uv$ for $x,y,u,v \in V$
QUESTION: $\exists$ tree T and map $s: V \to L(T)$ s.t.

possibly unrooted

$\forall\, xy|uv$: the shortest path from $s(x)$ to $s(y)$ in T disjoint from
———— " ———— $s(u)$ to $s(v)$ in T

<u>Remark</u> $(\mathbb{L};Q)$ is itself finitely bounded and its age is an amalgamation class $\rightsquigarrow$ D-relations

5.2. Branching-Time constraints and semilinear orders

Branching-Time satisfiability

INPUT: Finite relational structure $\mathbb{V}$ over $\{\leq, \|, \neq\}$

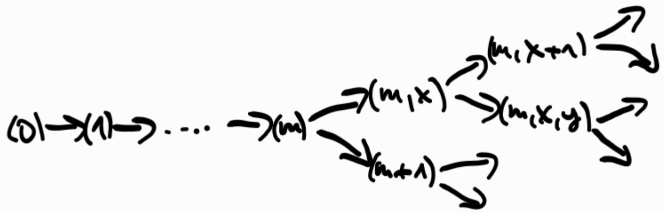
QUESTION: $\exists$ rooted tree $\mathbb{T}$ and map $s: V \to L(\mathbb{T})$:

- $x \leq_{\mathbb{V}} y \implies s(x)$ lies below $s(y)$
- $x \|_{\mathbb{V}} y \implies s(x)$ incomparable with $s(y)$
- $x \neq_{\mathbb{V}} y \implies s(x)$ and $s(y)$ not identical

5.2.1 An explicit construction

Def.: $\mathbb{S} := \bigcup_{n \geq 1} \mathbb{Q}^n$, for $a = (a_1, \ldots, a_m)$, $b = (b_1, \ldots, b_n) \in \mathbb{S}$, write:

- $a < b$ if ($m < n$ and $a_i = b_i \ \forall i \in [m]$) or ($a_i = b_i \ \forall i \in [m-1]$ and $a_m < b_m$)
- $a \leq b$ if $(a < b) \vee (a = b)$
- $a \| b$ if $(a = b) \vee (\neg(a \leq b) \wedge \neg(b \leq a))$



Remark $(\mathbb{S}; \leq, \|, \neq)$ ω-categorical, $\mathrm{Aut}(\mathbb{S}; \leq, \|, \neq)$ transitive

$CSP(\mathbb{S}; \leq, \|, \neq)$ is branching-time satisfiability

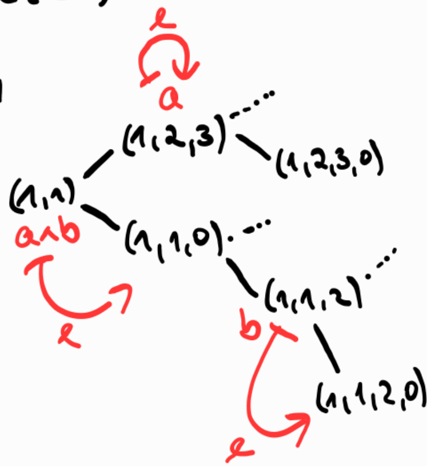
$(\mathbb{S}; \leq, \|, \neq)$ is <u>not</u> model-complete (to be checked!)

- $\phi_<(x,y,z) := z \leq x \wedge z \leq y \wedge \forall z'. (z' \leq x \wedge z' \leq y \Rightarrow z' \leq z)$

- $\forall x,y \in \mathbb{S}\ \exists z.\ \phi_<(x,y,z)$: (write $z = x \wedge y$)
  - if a and b comparable, then choose $z := \min(a,b)$
  - otherwise $a = (c, a_1, \dots, a_k)$, $b = (c, b_1, \dots, b_\ell)$ with $a_1 \neq b_1$, choose $z := (c, \min(a_1, b_1))$

- $a \| b \Rightarrow \exists$ embedding $e: (\mathbb{S}; \leq, \|, \neq) \hookrightarrow (\mathbb{S}; \leq, \|, \neq)$ s.t.:

$e|_{\mathbb{S} \setminus \{u \mid u \geq a \wedge b\}} := \mathrm{id}$, $e(a \wedge b, u') := (a \wedge b, u', 0)$



$(1,1) \leq (1,2,3)$ but $e(1,1) \not\leq e(1,2,3)$

$\Rightarrow e(a)\wedge e(b) = a\wedge b \neq (a\wedge b, 0) = e(a\wedge b)$ ■

<u>Remark</u> $(\mathbb{S}; \leq, \|, \neq)$ has ω-cat. model comp. (Theorem 4.6.4)

<u>5.2.2.</u> Construction via existential closure

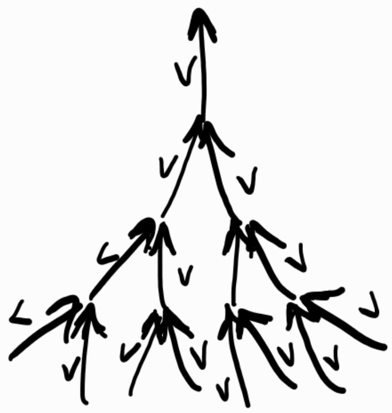
$T :=$ theory of semilinear orders

- Corollary 2.7.5: $\exists$ countable semilinear order $(\Pi; \leq)$ <u>ex.-closed</u> for T :⟺ embeds into a model of T & embeddings into models of T preserve complements of existential formulas
- clearly, $(\Pi; \leq)$ is:

:⟺ $x \leq y \wedge \neg(x = y)$

1) upwards directed: $\forall x, y\, \exists z\, (x \leq z \wedge y \leq z)$
2) dense: $\forall x, y\, (x < y \Rightarrow \exists z\, (x < z < y))$
3) unbounded: $\forall x\, \exists y, z\, (y < x < z)$

o.w. incompatible with 6) "without joins"



$:\Leftrightarrow \neg(u \leq x) \wedge \neg(x \leq u)$

4) binary branching: (a) $\forall x,y\,(x<y \Rightarrow \exists u(u<y \wedge u \| x))$

(b) $\forall x,y,z\,(x\|y \wedge x\|z \wedge y\|z \Rightarrow \exists u$ (larger than two out of three and incomparable to the third)

5) "nice": $\forall x,y\,(x\|y \Rightarrow \exists z\,(z>x \wedge z\|y))$

6) without joins: $\forall x,y,z\,(x \leq z \wedge y \leq z \wedge x\|y \Rightarrow$
$\exists u(x \leq u \wedge y \leq u \wedge u < z))$

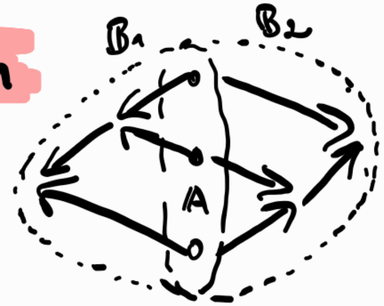
<u>Remark:</u> all ctble. semilinear orders satisfying (1)–(6) are isomorphic (back-and-forth)

$\Rightarrow (\mathbb{T};\leq)$ ω-categorical

<u>Remark:</u> $(\mathbb{T};\leq)$ model companion for T (Theorem 2.7.16)

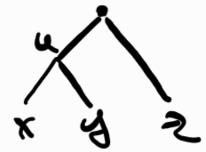
$CSP(\mathbb{T};\leq,\|,\neq)$ is Branching-Time Satisfiability

5.2.3 Construction via Fraïssé amalgamation



- recall that $\mathrm{Age}(\mathbb{T};\leq)$ does not have the AP

Def.: $xy|z \;:\Leftrightarrow\; \exists u\big((u\leq x)\wedge(u\leq y)\wedge\neg(u\leq z)\wedge\neg(z\leq u)\big)$



Prop. 5.2.2 $(\mathbb{T};\leq,|)$ is homogeneous. ■

Prop. 5.2.3 $(\mathbb{T};\leq,|)$ is finitely bounded.

Proof. • Write $x|y$ as shortcut for $\neg(x\leq y \vee y\leq x)$

- $\Phi :=$ universal axioms for semilinear orders $\wedge$:

$$\forall x,y,z,u: (u\leq x \wedge u\leq y \wedge u|z) \Rightarrow xy|z \tag{18}$$

$$\wedge\ \forall x,y,z: (xy|z \Rightarrow yx|z) \tag{19}$$

$$\wedge\ \forall x,y,z: (xy|z \Rightarrow x|z \wedge y|z) \tag{20}$$

$$\vdots$$

$\vdots$

$$\wedge \forall x,y,z\ \neg(xy|z \wedge yz|x) \tag{21}$$

$$\wedge \forall x,y,z\ (x|y \wedge y|z \wedge x|y) \Rightarrow (xy|z \vee yz|x \vee zx|y) \tag{22}$$

$$\wedge \forall x,y,z,u\ (xy|z \wedge yz|u) \Rightarrow xz|u \tag{23}$$

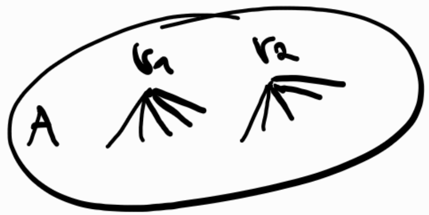
- clearly $\mathrm{Age}(\mathbb{T};\leq)$ satisfies above axioms
- $\mathbb{A}$ any finite model of Φ
  - clearly its $\{\leq\}$-reduct is a semilinear order
  - statement immediate if $|A|\leq 2 \Rightarrow$ assume $|A|>2$
  - proof by induction: assume that all proper substr. embed into $(\mathbb{T};\leq,|)$

<u>case 1</u> $\mathbb{A}\models \exists r\, \forall a\, (a\leq r)$

- assumption: $\exists e: \mathbb{A}[A\setminus\{r\}] \hookrightarrow (\mathbb{T};\leq,|)$
- upwards directed: $\exists t\in\mathbb{T}: t\geq e(a)\ \forall a\in A$
- $\mathbb{A}\models \forall x,y: \neg(rx|y \vee xr|y \vee xy|r)$, o.w. ↯ zu (20)

$\Rightarrow$ take the extension of e that maps r to t

<u>**case 2**</u> $\mathbb{A} \models \exists r_1, r_2 \left(r_1 | r_2 \land \forall a : \neg(a > r_1) \lor \neg(a > r_2) \right)$



- $(18) \Rightarrow ab|c \quad \forall a, b \in \downarrow r_i \,,\, c \in A \setminus \downarrow r_i$
- $(21) \Rightarrow \neg ab|c \quad \forall a \in \downarrow r_1, b \in \downarrow r_2, c \in \downarrow r_1 \cup \downarrow r_2$
- choose B_1, B_2 s.t. $|B_1| + |B_2|$ maximal and

$$\forall a, b \in B_i \, \forall c \in B_{3-i} \, (ab|c)$$

- suppose $\exists d \in A \setminus (B_1 \cup B_2)$
- if $d \geq b$ for some $b \in B_i$ $\overset{(18)}{\Longrightarrow} db|a \; \forall a \in B_{3-i}$

$\overset{(19)+(23)}{\Longrightarrow} dc|a \; \forall c \in B_i$

$\Longrightarrow \lightning$ to maximality (take $B_i \cup \{d\}$)

- similarly $d \leq b$ for some $b \in B_i \Longrightarrow \lightning$ to maximality

$\overset{(22)}{\Longrightarrow} \forall b_i \in B_i \; (b_1 b_2 | d \lor b_2 d | b_1 \lor d | b_1 b_2)$

$\overset{(23)}{\Longrightarrow} \lightning$ to maximality

- Inductive assumption: $\exists e_i : A[B_i] \hookrightarrow (\Pi_i \leq_i |)$
- upwards directed & unbounded $\Longrightarrow \exists u_1, u_2 \in \Pi \; (u_i > e_i(a) \; \forall a \in B_i)$
- homogeneity of $(\Pi_i \leq_i |)$: w.l.o.g. $u_1 | u_2$

$$\Longrightarrow e_1(a_1) \,|\, e_2(a_2) \quad \forall a_1 \in B_1, a_2 \in B_2$$

- choose e as the common extension of e_1 and e_2 $\blacksquare$