5.6 & 5.8:

Study CSPs definable by a sentence of a fixed logic
Given $\Phi \in$ FO / ESO / SNP / MMSNP we write

$$[\![\Phi]\!] = \{\mathcal{M} \in \text{FinRel} : \mathcal{M} \models \Phi\}$$

When $CSP(\mathcal{B}) = [\![\Phi]\!]$ we say that the CSP is described by Φ.

- What can we say about $\mathcal{B}$ when $CSP(\mathcal{B}) = [\![\Phi]\!]$?
- What choice of $\mathcal{B}$ implies that $\exists \Phi$ with $CSP(\mathcal{B}) = [\![\Phi]\!]$?

5.6.1: FO

Recall: Every CSP is equal to $\text{Forb}^{\text{Hom}}(\mathcal{F})$, where $\mathcal{F}$ contains finite connected structures.
Moreover, if $\mathcal{F}$ is finite then $\Phi := \bigwedge_{\mathcal{M} \in \mathcal{F}} \neg \psi_M$

in which case the problem is described by a $\forall^-$ sentence.

The converse is also true.

Theorem (B. Rossman) - Finite Homomorphism Preservation

Let τ be a fin. relational signature, and Φ an FO-sentence. Then Φ is equivalent to an $\exists^+$ sentence over FinRel(τ) iff $[\![\Phi]\!]$ is closed under homomorphisms.

Theorem (5.6.2)

Let τ be finite relational and $\mathcal{C} \subseteq \mathrm{FinRel}(\tau)$. TFAE:

1) $\mathcal{C} = \mathrm{CSP}(\mathcal{B})$ and $\exists \Phi \in \mathrm{FO}$ s.t. $\mathcal{C} = [\![\Phi]\!]$

2) $\mathcal{C} = \mathrm{Forb}^{\mathrm{Hom}}(\mathcal{F})$ for a finite set $\mathcal{F}$ of finite connected τ-structures.

3) $\mathcal{C} = \mathrm{CSP}(\mathcal{B})$ for an ω-categorical τ-structure $\mathcal{B}$ and $\exists \Psi \in \forall^{-}\mathrm{FO}$ s.t. $\mathcal{C} = [\![\Psi]\!]$.

Pf: $1 \Rightarrow 2$:

The class $\mathcal{D} = \{\mathcal{M} \in \mathrm{FinRel}(\tau) : \mathcal{M} \not\to \mathcal{B}\}$ is closed under homomorphisms. Since $\mathcal{D} = [\![\neg\Phi]\!]$, it follows by FHP that $\neg\Phi$ is equiv. over $\mathrm{FinRel}(\tau)$ to some $\exists^{+}$ sentence. So there is a $\forall^{-}$ sentence (in CNF) Ψ s.t.

$$\mathcal{C} = [\![\Psi]\!].$$

Pick such Ψ of minimal size. Let

$$\mathcal{F} = \{\mathcal{M} \text{ is the can. database of a conjunct of } \Psi\}$$

Clearly $\mathcal{F}$ is finite and $\mathcal{C} = \mathrm{Forb}^{\mathrm{hom}}(\mathcal{F})$. We argue that each $\mathcal{C} \in \mathcal{F}$ is connected. If not, then since $\mathcal{C} \not\to \mathcal{B}$ some c.c. $\mathcal{C}'$ of $\mathcal{C}$ satisfies $\mathcal{C}' \not\to \mathcal{B}$. But then Ψ was not minimal since the corresponding conjunct could have been by the conj. query of $\mathcal{C}'$. $\times$

$2 \Rightarrow 3$: Apply Theorem 4.3.8 to obtain an ω-categorical τ-structure $\mathcal{B}$ (w.out algebraicity) s.t. $\mathcal{C} = \mathrm{CSP}(\mathcal{B})$. Letting

$$\Psi := \bigwedge_{\mathcal{C} \in \mathcal{F}} \neg \psi_{\mathcal{C}}$$

we see that $\mathcal{C} = [\![\Psi]\!]$.

$3 \Rightarrow 1$: trivial. $\square$

5.6.2: Monadic SNP

Recall:

$\underset{\text{connected}}{C}\,\underset{\text{monotone}}{M}\,\underset{\text{monadic}}{M}$SNP: $\quad \exists \underbrace{P_1 \dots P_n}_{\text{unary}} \, \forall x_1 \dots x_k \; \psi$ ← q.f. in CNF

each τ-literal is negative
+ each clause is connected

Recall (Corollary 1.4.19): A formula $\Phi \in$ MSNP is such that $[\![\Phi]\!] = CSP(\mathcal{B})$ for some $\mathcal{B}$ $\Leftrightarrow$ $\Phi \equiv \Psi \in$ CMMSNP.

Theorem (5.6.3)
Fix finite rel. signature τ, and $\mathcal{C} \in Rel(\tau)$. If $CSP(\mathcal{C}) = [\![\Phi]\!]$ for some monadic SNP sentence Φ, then $\exists$ ω-categorical $\mathcal{B}$ s.t. $CSP(\mathcal{B}) = CSP(\mathcal{C})$.

Pf: By Corollary 1.4.19 we may assume wlg that Φ is in CMMSNP. Let $P_1, \dots, P_k$ be the $\exists$-quantified predicates in Φ, and

$$\tau' = \tau \cup \{P_1, \dots, P_k\} \cup \{P_1', \dots, P_k'\}.$$

Consider Φ' where we replace positive literals of the form $P_i(x)$ by $\neg P_i'(x)$. Then for every clause ψ of Φ', the formula $\neg\psi$ is q.f. p.p.
Consider

$$\mathcal{F} = \{\text{canonical database } D_{\neg\psi} \text{ of each } \neg\psi \text{ (as a } \tau'\text{-structure)}\}$$

Then:

- Φ connected $\Rightarrow$ every $\mathcal{C} \in \mathcal{F}$ is connected.
- $\forall$ τ'-structure $\mathcal{A}'$, and clause ψ of Φ': $\quad (*)$

$$\mathcal{A}' \models \psi \Leftrightarrow D_{\neg\psi} \not\to \mathcal{A}'.$$

We then obtain (Th. 4.3.8) an $\mathcal{F}$-free ω-categorical τ'-structure that is universal for all $\mathcal{F}$-free structures. Call this $\mathfrak{B}'$. Let

$$d(x) = \bigwedge_{i=1}^{k} \left(P_i(x) \oplus P_i'(x) \right)$$

Then the τ-reduct $\mathfrak{B}$ of $\mathfrak{B}'$ on the domain $d(\mathfrak{B}')$ is ω-categorical. We claim that $[\![\Phi]\!] = CSP(\mathfrak{B})$.

- $CSP(\mathfrak{B}) \subseteq [\![\Phi]\!]$:

  Take A finite with $A \xrightarrow{h} \mathfrak{B}$. Expand A into a τ'-structure A' so that $\forall i \in [k], \forall a \in A$:

$$A' \models P_i(a) \iff \mathfrak{B}' \models P_i(h(a))$$
$$A' \models P_i'(a) \iff \mathfrak{B}' \models P_i'(h(a)).$$

  Then h extends to a homomorphism $A' \to \mathfrak{B}'$.

  $\Rightarrow A' \in Forb^{hom}(\mathcal{F})$

  $\Rightarrow A' \models \psi$. for each clause of Φ' (by $*$)

  $\Rightarrow A \models \Phi$.

- $[\![\Phi]\!] \subseteq CSP(\mathfrak{B})$:

  Take A finite with $A \models \Phi$. Then there exist a τ'-exp. A' of A that satisfies the fo part of Φ' and such that for every $a \in A$ exactly one of $P_i(a)$ or $P_i'(a)$ holds.

  $\Rightarrow A' \models \psi$ for every clause of Φ'

  $\Rightarrow \forall C \in \mathcal{F}: C \not\to A'$

  $\Rightarrow A' \leq B'$ by universality of $\mathfrak{B}'$ wrt $Forb^{hom}(\mathcal{F})$

  $\Rightarrow A \leq B$ as $A' \models \forall x\, d(x)$

  $\Rightarrow A \in CSP(\mathfrak{B})$ $\square$

$\boxed{\text{The assumption that } \Phi \text{ is monadic is } \underline{\text{necessary}}.}$

Pf:

$$\Phi := \exists E, S, T\ \forall x, x', y, z\ \overbrace{\Big[E \text{ is an eq. relation} \wedge S \text{ extends Succ} \wedge T \text{ is } \overset{\text{total}}{\text{transitive}} + \text{irreflexive} + \text{extends } S \wedge \big(S(x,y) \wedge S(x,z) \rightarrow E(y,z)\big) \wedge \big(S(x,y) \wedge E(x,x') \rightarrow S(x',y)\big) \Big]}^{\varphi}$$

We claim that $[\![\Phi]\!] = CSP(\mathbb{Z}, Succ)$.

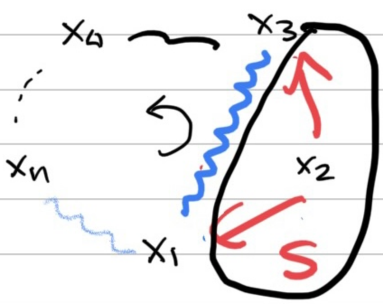
- $CSP(\mathbb{Z}, Succ) \subseteq [\![\Phi]\!]$:
  If $(G, Succ^G) \xrightarrow{h} (\mathbb{Z}, Succ)$ then expand G to a $\{Succ, E, S, T\}$-structure G' so that $\forall x,y \in G'$
    - $G' \models E(x,y) \iff h(x) = h(y)$
    - $G' \models S(x,y) \iff \mathbb{Z} \models Succ(h(x), h(y))$
    - $G' \models T(x,y) \iff h(x) < h(y)$

  Then trivially $G' \models \varphi \Rightarrow G \models \Phi$.
- $[\![\Phi]\!] \subseteq CSP(\mathbb{Z}, Succ)$:
  Let G' be a $\{Succ, E, S, T\}$-structure satisfying φ. Then it's $\{Succ\}$-reduct, G, is a DAG. (since T is a transitive + irreflexive extension of $Succ$).

  NTS: For every Succ-cycle C in G, $\# C = 0$ forward edges − backward edges

  STS: For every $\{Succ, S, E\}$-cycle C in G, $\# C = 0$ only count Succ & S edges

  Assume for $\times$ that there is such C with $\# C \neq 0$. Take such C of minimal length s.t. $\# C > 0$.



Consequently $G \rightarrow (\mathbb{Z}, Succ)$ □

Proposition (5.8.2) $CSP(\mathbb{Z}, Succ)$ cannot be formulated by an ω-categorical template

Pf: For every $n \in \mathbb{N}$ the formula

$$\varphi_n(x_0, x_n) := \exists x_1 \dots x_{n-1} \bigwedge_{i=1}^{n} Succ(x_{i-1}, x_i)$$

is in a different pp 2-type.

So

$$|S_2^{pp}(\mathbb{Z}, Succ)| = \aleph_0$$

We conclude by Corollary 4.6.2.

5.6.3: Guarded MSNP

$$\exists R_1 \dots R_k \, \forall x_1 \dots \forall x_\ell \; \varphi$$

arbitrary predicates over signature ρ

CNF with negative τ-literals

every conjunct is of the form

$$\underbrace{\alpha_1 \wedge \dots \wedge \alpha_n}_{\text{atomic } (\tau \cup \rho) \text{ formulas "body atoms"}} \rightarrow \underbrace{\beta_1 \vee \dots \vee \beta_m}_{\text{atomic } \rho\text{-formulas "head atoms"}}$$

+ for every head atom β_i there a body atom α_j s.t. α_j contains all variables from β_i.

5.6.5
Prop: Every $\Phi \in$ GMSNP is equivalent to a finite disjunction $\Phi_1 \vee \dots \vee \Phi_k$ of connected GMSNP formulas

Theorem (5.6.6): For every Φ in connected GMSNP there exists a reduct C of a finitely bounded homogeneous structure s.t. $[\![\Phi]\!] = CSP(C)$.