§1.1 HOMOMORPHISM PERSPECTIVE

τ - RELATIONAL SIGNATURE (usually finite)

$\mathbb{A}$ - RELATIONAL τ-STRUCTURE with domain A and relations for the rel symbols of τ

A HOMOMORPHISM of τ-STRUCTURES $\mathbb{A}, \mathbb{B}$ is a function

$$h : A \to B \quad \text{s.t.} \quad \underset{(a_1, \dots, a_n)}{\underset{\parallel}{\bar{a}}} \in R^{\mathbb{A}} \Rightarrow \underset{(h(a_1), \dots, h(a_n))}{\underset{\parallel}{h(\bar{a})}} \in R^{\mathbb{B}}$$

CSP($\mathbb{B}$) is the computational problem of deciding whether a given finite τ-structure $\mathbb{A}$ maps hom to $\mathbb{B}$

EXAMPLES

- $CSP(\mathbb{Z}, <)$

A $\{<\}$-structure can be represented as a DIGRAPH

$a \longrightarrow b$ iff $a < b$

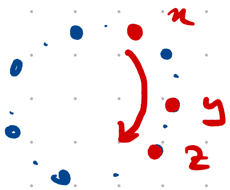
A has no directed cycles iff $<$ extends to a total order on A iff $A \in CSP(\mathbb{Z}, <)$

DEPTH-FIRST ALGORITHM it runs in linear time

$CSP(\mathbb{Z}, <) \in P$.

- $CSP(\mathbb{Z}, cyc)$

$cyc(x,y,z)$ iff $(x<y<z) \vee (y<z<x) \vee (z<x<y)$



GALIL & MEGIDDO (1997) NP COMPLETE (reduce 3SAT to this problem)

- $CSP(K_k)$ $\quad G \xrightarrow{hom} K_k$. This is equivalent: is G k-colourable?

$k=2 \in P$ $\quad k \geq 3$ NP-complete (transformed from 3SAT KARP 1972)

B is CONNECTED if it is NOT the disjoint union of two non-empty structures.

A maximal connected subset of B is a conn. component

$\mathcal{C}$ class of fin rel struc

CLOSED UNDER HOMOMS if $\forall A \in \mathcal{C}$ $A \xrightarrow{hom} B \Rightarrow B \in \mathcal{C}$

// INVERSE HOM $\forall B \in \mathcal{C}$ $A \xrightarrow{hom} B \Rightarrow A \in \mathcal{C}$

// DISJOINT UNIONS

$\mathcal{F}$ of τ-structs.

A is $\mathcal{F}$-free if no $B \in \mathcal{F}$ maps hom to A

$\mathrm{Forb}^{hom}(\mathcal{F})$ class of finite $\mathcal{F}$-free structures.

EQUIV to CSP τ fin. rel. sig. $\mathcal{C}$ a class of fin τ-struct. TFAE:

1. $\mathcal{C} = CSP(B)$ for some τ-struct B
2. $\mathcal{C} = Forb^{hom}(\mathcal{F})$ for $\mathcal{F}$ a class of fin. connected τ-structs
3. $\mathcal{C}$ is closed under disj. unions & inverse homs. $\circledast$
4. $\mathcal{C} = CSP(B)$ for some ctable τ-structure B.

Proof: (1) $\Rightarrow$ (2) $\mathcal{F} :=$ finite conn τ-structures not mapping hom to B

$CSP(B) \subseteq Forb(\mathcal{F})$: $A^{\in \mathcal{F}} \longrightarrow D \longrightarrow B$ then $A \longrightarrow B$ $\times$

If $A \not\to B$ then some conn comp A' of A $\exists$ s.t. $A' \not\to B$

$A' \in \mathcal{F}$ and $A' \xrightarrow{hom} A$ so $A \notin Forb(\mathcal{F}) \Rightarrow Forb(\mathcal{F}) \subseteq CSP(B)$.

(2) $\Rightarrow$ (3): $Forb(\mathcal{F})$ is closed under disj unions & inverse homs.

(3) $\Rightarrow$ (4): $\mathcal{C}' \subseteq \mathcal{C}$ with one structure per isom type.

Let $B = \biguplus_{C \in \mathcal{C}'} C$

- $C \in \mathcal{C} \Rightarrow C \xrightarrow{hom} B$
- Say $A \xrightarrow{h} B$. Then $h(A) \subseteq D^{*}$

$*$ D is the disj. union of structures from $\mathcal{C}$.

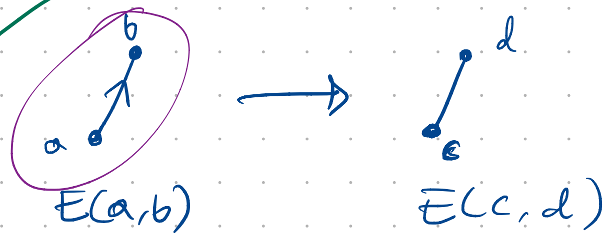
USE (3)

so $CSP(B) = \mathcal{C}$. $\blacksquare$

1.1.9 G finite graph.

Q: Is G Δ-free?

CSP(B) for some $\{E, E\}$-structure



B has to be infinite

B is a CORE if all endomorphisms are embeddings.

$B \xrightarrow{\text{hom}} B$

$f: A \to B$ INJECTIVE

$\bar{a} \in R^A \iff f(\bar{a}) \in R^B$

If A is finite it has a CORE which is hom eq. to it (on unique up to isom)

$A \underset{\text{hom}}{\overset{\text{hom}}{\rightleftarrows}} B$ CSP(A) and CSP(B) have the same complexity.

Proof: suppose A not a core. $f: A \to A$ end but not an embedding.

$|f(A)| = |A|$
- f is inj.
- f is an embedding. $f(\bar{a}) \in R^A$ but $\bar{a} \notin R^A$. ↯

$|f(A)| < |A|$. A and f(A) are hom eq. $A \underset{\text{id}}{\overset{f}{\rightleftarrows}} f(A)$

By induction on |A| this process terminates ∎

§ 1.2 SENTENCE EVALUATION PERSPECTIVE

τ-formula $\phi(x_1, \dots x_n)$ is PRIMITIVE POSITIVE if

INSTANCE

$$\exists x_{n+1} \dots x_m \, (\underbrace{\psi_1}_{} \wedge \dots \wedge \underbrace{\psi_\ell}_{})$$

CONSTRAINTS

atomic formulas or $\perp$ or T.

CSP(B) Given a pp-sentence ϕ, is ϕ true in B

SOLUTION $f : V \to B$ s.t. $B \models \bigwedge \psi_i (f(\bar{v}))$

vars of ϕ

$\psi_1 \wedge \dots \wedge \psi_\ell$ is satisfiable?

CANONICAL CONJ. QUERIES

$A = (a_1 \dots a_n)$

treating a_is as variables

$$Q(A) = \bigwedge_{\substack{R \in \tau \\ (a_1 \dots a_k) \in R^A}} R(a_1 \dots a_k)$$

$A \xrightarrow{hom} B$ iff $Q(A)$ is satisf in B.

CANONICAL DATABASES

ϕ a pp-formula — <u>without</u> $=$ or $\perp$

$D(\phi)$ has domain the variables of ϕ

$(v_1 \dots v_r) \in R^{D(\phi)}$ iff $R(v_1 \dots v_r)$ appears in the ψ_i

ϕ is TRUE IN B iff $D(\phi) \xrightarrow{hom} B$

$Q(D(\phi)) = \phi$ $\quad D(Q(A)) = A$.

J, C, G (1997) τ is finite $\mathbb{B}$ is a τ-structure.
R has a pp-definition in $\mathbb{B}$. Then
CSP($\mathbb{B}$) and CSP($\mathbb{B}$, R) are linear time eq.

in finite core structures orbits of k-tuples are pp-def. so we can add constants

REMOVING "TUPLES" RELS $R = \{(b_1, \dots, b_k)\}$ suppose orbit of $\bar{b}$ under Aut($\mathbb{B}$) is pp-def. Then, there is a poly time reduction from CSP($\mathbb{B}$, R) to CSP($\mathbb{B}$).

proof. ϕ instance of CSP($\mathbb{B}$, R).

STEP 1: If ϕ has multiple instances of $R(x_1 \dots x_k)$ $R(y_1 \dots y_k)$
REPLACE VARIABLES so that we only have one instance $R(x_1 \dots x_k)$

STEP 2: REPLACE $R(x_1 \dots x_k)$ with pp-def of Aut($\mathbb{B}$)-orbit of $\bar{b}$.

ψ in τ pp.

ϕ is true in ($\mathbb{B}$, R) iff ψ is true in $\mathbb{B}$.

($\Rightarrow$)

($\Leftarrow$) suppose $s': V_\psi \longrightarrow \mathbb{B}$ is a sol to ψ.

$\mathbb{B} \models \theta(s(\bar{x}))$. $\underbrace{\text{Take } \alpha \in \text{Aut}(\mathbb{B}) \text{ s.t. } \alpha(s(\bar{x})) = \bar{b}}$

pp def of the orbit of $\bar{b}$

$\mathbb{B} \models \psi'(\alpha(s'(\bar{v})))$. So can extend this assignment to $s: V_\phi \longrightarrow \mathbb{B}$

conj. of the constraints of ψ s.t. $\mathbb{B} \models \phi'(s(\bar{v}'))$

conj. of constraints of ϕ. ☒