CSP Reading Group Sections 6.1.1–6.1.3

B set, $n \geq 1$ $\qquad \mathcal{O}_B^{(n)} = B^{B^n} = \{\text{maps } B^n \to B\}$ ... n-ary operations on B

$$\mathcal{O}_B = \bigcup_{n\geq 1} \mathcal{O}_B^{(n)}$$ ... operations on B

operation clone : $\mathcal{C} \subseteq \mathcal{O}_B$ satisfying

↓ sometimes called "concrete clone"

- $\mathcal{C}$ contains all projections π_i^n $\forall n$ $\forall i \in \{1, \dots, n\}$
- $\mathcal{C}$ closed under composition, i.e.

$\forall f \in \mathcal{C}$ n-ary and $\forall g_1, \dots, g_n \in \mathcal{C}$ k-ary

$f(g_1, \dots, g_n) \in \mathcal{C}$ ⇝ this gives in fact every composition.

$(x_1, \dots, x_k) \mapsto f(g_1(x_1, \dots, x_k), \dots, g_n(x_1, \dots, x_k))$

$\mathcal{Y} \subseteq \mathcal{O}_B \rightsquigarrow \langle \mathcal{Y} \rangle$ ... smallest clone that contains $\mathcal{Y}$

DEF.: polymorphism of a structure $\mathbb{B}$ is a homomorphism from a finite power of $\mathbb{B}$ to $\mathbb{B}$

$\mathrm{Pol}(\mathbb{B})$ = the $\underbrace{\text{clone}}_{\text{easy to verify}}$ of all polymorphisms of $\mathbb{B}$

$f \in \mathcal{O}_B^{(n)}$ preserves $R \subseteq B^k$ if $\forall r_1, \dots, r_n \in R$

$\Updownarrow$

(R is invariant under f)

$$f\left(\underset{\substack{r_1 \\ \in \\ R}}{\begin{pmatrix} \\ \\ \\ \end{pmatrix}} \cdots \underset{\substack{r_n \\ \in \\ R}}{\begin{pmatrix} \\ \\ \\ \end{pmatrix}}\right) = \begin{pmatrix} f(\dots) \\ \vdots \\ f(\dots) \end{pmatrix} \in R$$

OBSERVATION: f is a polymorphism of $\mathbb{B}$ $\Leftrightarrow$ f preserves all relations of $\mathbb{B}$

DEF.: operation clone $\mathscr{C} \subseteq \mathscr{O}_B$ is finitely related if
$\exists$ structure $\mathbb{B}$ with finite relational signature
s.t. $\mathrm{Pol}(\mathbb{B}) = \mathscr{C}$

$$R_B = \bigcup_{n \geq 1} R_B^{(n)} \quad \dots \quad \text{(n-ary) relations on } B$$

NOTATION: $R \subseteq R_B \rightsquigarrow \mathrm{Pol}(R)$

$\mathcal{Y} \subseteq \mathscr{O}_B \rightsquigarrow \mathrm{Inv}(\mathcal{Y})$

PROPOSITION: For every $\mathbb{B}$ $\underbrace{\langle \mathbb{B} \rangle_{pp}}_{\text{relations definable by a pp-formula}} \subseteq \mathrm{Inv}(\mathrm{Pol}(\mathbb{B}))$

Proof: $R \subseteq B^k$, $f \in \mathrm{Pol}(\mathbb{B})$ ℓ-ary
pp-def. of R $\psi(x_1, \dots, x_k) = \exists x_{k+1}, \dots, x_n \ \psi'(x_1, \dots, x_{k+1}, x_k, \dots, x_n)$
$t_1, \dots, t_\ell \in R \rightsquigarrow s_1, \dots, s_\ell$ extended by witnesses so that they satisfy
$\psi' \Rightarrow f(s_1, \dots s_\ell)$ satisfies $\psi' \Longrightarrow f(t_1, \dots, t_\ell) \in R$ $\blacksquare$

6.1.1 Pol-Inv

B ... countably infinite

DEF.: $\mathcal{Y} \subseteq \mathcal{O}_B$ is *locally closed* $\Leftrightarrow$

$[\forall n \geq 1 \ \forall g \in \mathcal{O}_B^{(n)} : (\forall F \subseteq B \text{ finite } \ \exists f \in \mathcal{Y} \ \ g|_F = f|_F) \Rightarrow g \in \mathcal{Y}]$

$\overline{\mathcal{Y}}$ ... *local closure* of $\mathcal{Y}$ = smallest locally closed subset containing $\mathcal{Y}$

$f \in \overline{\langle \mathcal{Y} \rangle}$ ... f is *locally generated* by $\mathcal{Y}$

PROPOSITION: $\mathcal{C} \subseteq \mathcal{O}_B$ ... the smallest locally closed clone that contains $\mathcal{Y} \subseteq \mathcal{O}_B \Rightarrow \mathcal{C} = \overline{\langle \mathcal{Y} \rangle} = \mathrm{Pol}(\mathrm{Inv}(\mathcal{Y}))$.

PROPOSITION: $\mathscr{C} \subseteq \mathcal{O}_B$ ... the smallest locally closed clone that contains $\mathscr{G} \subseteq \mathcal{O}_B$ $\Rightarrow$ $\mathscr{C} = \overline{\langle \mathscr{G} \rangle} = \mathrm{Pol}(\mathrm{Inv}(\mathscr{G}))$.

Proof:

$\overline{\langle \mathscr{G} \rangle} \subseteq \mathscr{C}$ is clear because $\mathscr{C}$ is a locally closed clone

For $\mathscr{C} \subseteq \overline{\langle \mathscr{G} \rangle}$ we show that $\overline{\langle \mathscr{G} \rangle}$ is a locally closed clone

(since $\mathscr{G} \subseteq \overline{\langle \mathscr{G} \rangle}$).

$\overline{\langle \mathscr{G} \rangle}$ is locally closed and contains projections.

$\overline{\langle \mathscr{G} \rangle}$ is closed under composition: $f, g_1, \dots, g_n \in \overline{\langle \mathscr{G} \rangle}$, $F \subseteq B$ finite

$\rightsquigarrow$ approximate $g_1, \dots, g_n$ on F by $g_1', \dots, g_n' \in \langle \mathscr{G} \rangle$ and f by f' on $g_1(F) \cup \dots g_n(F)$, then $f'(g_1', \dots, g_n')|_F = f(g_1, \dots, g_n)|_F$

$\Rightarrow \overline{\langle \mathscr{G} \rangle} = \mathscr{C}$

$\overline{\langle \mathcal{Y} \rangle} \subseteq \mathrm{Pol}(\mathrm{Inv}(\mathcal{Y}))$:

$f \in \overline{\langle \mathcal{Y} \rangle}$ n-ary, $t_1, \dots, t_n \in R$

Find $g \in \langle \mathcal{Y} \rangle$ s.t. $f(t_1, \dots, t_n) = g(t_1, \dots, t_n)$.

$\mathrm{Inv}(\mathcal{Y}) = \mathrm{Inv}(\langle \mathcal{Y} \rangle) \Rightarrow g \in \langle \mathcal{Y} \rangle \subseteq \mathrm{Pol}(\mathrm{Inv}(\langle \mathcal{Y} \rangle)) = \mathrm{Pol}(\mathrm{Inv}(\mathcal{Y}))$

$\Rightarrow f(t_1, \dots, t_n) = g(t_1, \dots, t_n) \in R$

$\mathrm{Pol}(\mathrm{Inv}(\mathcal{Y})) \subseteq \overline{\langle \mathcal{Y} \rangle}$:

$f \in \mathrm{Pol}(\mathrm{Inv}(\mathcal{Y}))$ n-ary, $F \subseteq B$ finite.

$t_1, \dots, t_m$ ... all elements of F^n

$R := \{ \underbrace{(g(t_1), \dots, g(t_m))}_{m\text{-tuple}} \mid g \in \langle \mathcal{Y} \rangle^{(n)} \}$ is in $\mathrm{Inv}(\mathcal{Y})$

$\Rightarrow f$ preserves R

$\pi_i^n \in \langle \mathcal{Y} \rangle^{(n)} \ \forall i \Rightarrow (t_1[i], \dots, t_m[i]) \in R \ \forall i \Rightarrow f(t_1, \dots, t_m) \in R$

$\Rightarrow (f(t_1), \dots, f(t_m)) = (g(t_1), \dots, g(t_m))$ for some $g \in \langle \mathcal{Y} \rangle \Rightarrow f|_F = g|_F \ \blacksquare$

COROLLARY: For $\mathcal{C} \subseteq \mathcal{O}_B$ TFAE:

(1) $\mathcal{C}$ is the polymorphism clone of a relational structure

(2) $\mathcal{C}$ is a locally closed clone

Proof:

(1) $\Rightarrow$ (2): $\mathcal{C} = \mathrm{Pol}(\mathbb{A})$ ... we have seen that these are locally closed and clones

(2) $\Rightarrow$ (1): $\mathcal{C} = \overline{\langle \mathcal{C} \rangle} = \mathrm{Pol}(\underbrace{\mathrm{Inv}(\mathcal{C})}_{\text{gives rise to a structure on } B})$ $\blacksquare$

Locally closed subclones of $\mathcal{O}_B$ form a complete lattice:

$$(\mathcal{C}_i)_{i\in I} \xrightarrow{\text{join}} \overline{\left\langle \bigcup_{i\in I} \mathcal{C}_i \right\rangle}$$

$$(\mathcal{C}_i)_{i\in I} \xrightarrow{\text{meet}} \bigcap_{i\in I} \mathcal{C}_i$$

DEF: $\mathcal{G}$ ... a permutation group on the set B

We write $\mathrm{Orb}(\mathcal{G})$ for a relational structure on B whose relations are orbits of n-tuples of $\mathcal{G}$ $\forall n \geq 1$.

typo in the book (I think)

Some properties:

- $\mathrm{Pol}(\mathrm{Orb}(\mathcal{G}))$ = the unique largest clone $\mathscr{C}$ s.t. $\mathscr{C}^{(1)} = \overline{\mathcal{G}}$
- $\mathrm{End}(\mathrm{Orb}(\mathcal{G})) = \overline{\mathcal{G}}$
- $\mathcal{G}$ oligomorphic $\Rightarrow$ (fo-expansions of) $\mathrm{Orb}(\mathcal{G})$ are mc-cores

REMARK: $\mathrm{Pol}(\mathrm{Orb}(\mathcal{G}))^{(n)}$ consists precisely of operations $f \in \mathcal{O}_B^{(n)}$ s.t. $\forall g_1, \dots, g_n \in \mathcal{G}$

$$x \longmapsto f(g_1(x), \dots, g_n(x)) \text{ is in } \overline{\mathcal{G}}$$

6.1.2 Inv-Pol

DEF.: infinitary pp-formulas are defined inductively:

- atomic formulas
- finite and infinite conjunctions of formulas $\phi(x_1, \dots, x_n)$ in the class with the same free variables $x_1, \dots, x_n$
- $\exists x_i \, \phi(x_1, \dots, x_n)$ where $\phi(x_1, \dots, x_n)$ is in the class

EXAMPLE: $\mathbb{B} = (\mathbb{Z} \cup \{\infty\}, <)$

$\forall i \in \mathbb{N}$ $\{(n,m) \in B^2 \mid n+i<m\}$ is pp-definable by $\phi_i(x,y)$.

$\{\infty\}$ by $\exists x \bigwedge_{i=1}^{\infty} \phi_i(x,y)$.

pp-formulas satisfiable by ∞ are always satisfiable by $z \in \mathbb{Z}$ $\Rightarrow$ all elements of $\mathbb{Z}$ $\Rightarrow$ infinite conjuctions of pp-formulas always define a relation containing $\mathbb{Z}$

chain : $(R_i)_{i\in\mathbb{N}}$ all R_i of the same arity and $R_i \subseteq R_{i+1}$ $\forall i$

THEOREM: $\mathbb{B}$ countable

$R \subseteq B^n$ is preserved by $\mathrm{Pol}(\mathbb{B}) \Longleftrightarrow$ R is the union of a chain of relations that have infinitary pp-definitions in $\mathbb{B}$

Proof:

Easy to check that every such R is preserved by $\mathrm{Pol}(\mathbb{B})$ (we have already seen for classical pp-definitions).

$R \in \mathrm{Inv}(\mathrm{Pol}(\mathbb{B}))$ of arity n.

let $a_1, a_2, \ldots$ be an enumeration of R.

$R_i :=$ intersection of all relations with infinitary pp-definitions in $\mathbb{B}$ containing $a_1, \ldots, a_i$

$R_i \subseteq R_{i+1}$ $\forall i$

We show $R = \bigcup_{i \in \mathbb{N}} R_i$.

$R \subseteq \bigcup_{i \in \mathbb{N}} R_i$: Clear ($B^n$ is always in the intersections).

To prove $\supseteq$, we show $R_i \subseteq R \ \forall i$, let $t = (t_1, \dots, t_n) \in R_i$.

We search for $\underline{f \in \mathrm{Pol}(\mathbb{B})}$ of arity i s.t. $\underline{f(a_1 | \dots | a_i) = \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}}$.

$\underbrace{b_1, b_2, \dots \text{ enumeration of } B^i}$ s.t.

$$\begin{array}{c} b_1 \rightarrow \\ b_2 \rightarrow \\ \vdots \\ b_n \rightarrow \\ \vdots \end{array} \left(\begin{array}{c|c|c|c} a_1 & a_2 & \dots & a_i \\ & \vdots & & \end{array} \right)$$

WLOG no rows repeat, otherwise we find a definition of a projection of R and then pp-define R from it by equalities

Define $f(b_1) = t_1, \dots, f(b_n) = t_n$ $\rightsquigarrow$ this partial definition preserves all infinitary pp-formulas since $t \in R_i$

Extend f already defined on $b_1, \dots, b_m \in B'$, $m \geq n$:

Take $\phi(\bar{x}_1, \dots, \bar{x}_{m+1})$ conjuction of all infinitary pp-formulas satisfied by $b_1, \dots, b_{m+1}$

f preserves $\phi'(\bar{x}_1, \dots, \bar{x}_m) = \exists \bar{x}_{m+1} : \phi(\bar{x}_1, \dots, \bar{x}_{m+1})$

$\Rightarrow \mathbb{B} \models \phi'(f(b_1), \dots, f(b_m)) \rightsquigarrow t_{m+1}$ witness for ϕ'

$f(b_{m+1}) := t_{m+1}$

this extension preserves every infinitary pp-formula over $\mathbb{B}$

in particular, $f \in \mathrm{Pol}(\mathbb{B}) \Rightarrow$

$$\underset{\underset{R_i}{\in}}{t} = \begin{pmatrix} f(b_1) \\ \vdots \\ f(b_n) \end{pmatrix} = f\left(\underbrace{a_1}_{\in R} \Big| \dots \Big| \underbrace{a_i}_{\in R}\right) \in R$$

$\Rightarrow R_i \subseteq R \overset{i \text{ arbitrary}}{\Longrightarrow} R = \bigcup_{i \in \mathbb{N}} R_i$ as we wanted $\blacksquare$

6.1.3 Oligomorphic clones

$\mathcal{C} \subseteq \mathcal{O}_B$ operation clone

$e \in \mathcal{C}^{(1)}$ is invertible if $\exists i \in \mathcal{C}^{(1)}: \forall x \in B \quad e(i(x)) = i(e(x)) = x$

$\mathcal{C} = \mathrm{Pol}(\mathbb{B}) \Rightarrow$ invertible operations in $\mathcal{C}$ are $\mathrm{Aut}(\mathbb{B})$

DEF.: $\mathcal{C} \subseteq \mathcal{O}_B$ ... operational clone

$\mathcal{C}$ is oligomorphic if the set of invertible operations in $\mathcal{C}$ forms an oligomorphic permutation group.

THM 4.1.6 + COR 6.1.6 imply:

locally closed clone $\mathcal{C} \subseteq \mathcal{O}_B$ is oligomorphic

$\Updownarrow$

$\mathcal{C} = \mathrm{Pol}(\mathbb{B})$, $\mathbb{B}$ ω-categorical

THEOREM: $\mathbb{B}$ countable ω-categorical

$$\underbrace{\langle \mathbb{B} \rangle_{pp}}_{\text{relations pp-definable in } \mathbb{B}} = \mathrm{Inv}(\mathrm{Pol}(\mathbb{B}))$$

Proof: $\subseteq$ we already know

$\supseteq$: For each fixed arity ... only finitely many pp-definable relations

$\Rightarrow$ infinite conjunctions of pp-formulas are equivalent to pp-formulas

$\Rightarrow$ unions of chains of pp-definable relations are pp-def.

THEOREM $\Rightarrow$ $\forall R \in \mathrm{Inv}(\mathrm{Pol}(\mathbb{B}))$ is pp-definable

$\blacksquare$

THEOREM: $\mathbb{B}$ countable, ω-categorical

(1) sets $\langle \mathfrak{C} \rangle_{pp}$, $\mathfrak{C}$ fo-reducts of $\mathbb{B}$, form a lattice

(2) closed subclones of $\mathcal{O}_B$ containing $\mathrm{Aut}(\mathbb{B})$ form a lattice

(3) operators Inv and Pol are mutually inverse anti-isomorphisms of these lattices

COROLLARY: $\mathbb{B}$ ω-categorical, $\mathcal{F} \subseteq \mathcal{O}_B$

- $\overline{\langle \mathcal{F} \rangle} = \mathrm{Pol}(\mathbb{B})$ iff $\langle \mathbb{B} \rangle_{pp} = \mathrm{Inv}(\mathcal{F})$
- the smallest relation in $\langle \mathbb{B} \rangle_{pp}$ containing $R \subseteq B^k$ is

$$\{ f(a_1, \dots, a_n) \mid n \in \mathbb{N}, f \in \mathrm{Pol}^{(n)}(\mathbb{B}), a_1, \dots, a_n \in R \}$$

EXAMPLE: $\mathbb{B} = (\mathbb{Z}, \{(x,y) \mid x = y+1\})$

$\mathrm{Inv}(\mathrm{Pol}(\mathbb{B})) = \langle \mathbb{B} \rangle_{pp}$ (even though $\mathbb{B}$ is not ω-cat.)

We show $\langle \mathbb{B} \rangle_{pp}$ is closed under infinite conjunctions and unions of chains.

Fact: Every relation is a conjunction of binary relations.

Binary relations: $\emptyset, \mathbb{Z}^2, \{(x,y) \mid x = y + c\}, c \in \mathbb{Z}$

↳ infinite conjunctions of these are of the same form

↳ unions of such chains are of this form