§ 6.1.4 ESSENTIALLY UNARY OPERATIONS

Let $k \in \mathbb{N}$, $i \in \{1, \dots, k\}$

For $f \in \mathcal{O}_B^{(k)}$, we say the ith argument is fictitious if $\exists f' \in \mathcal{O}_B^{(k-1)}$ s.t. $f(x_1 \dots x_k) \approx f'(x_1 \dots x_{i-1}\, x_{i+1} \dots x_k)$.

If the ith argument is NOT fictitious, we say

f DEPENDS ON THE ith argument

This equivalent to $\exists x_1 \dots x_k\, x_i' \in B$ s.t. $f(x_1 \dots x_k) \neq f(x_1 \dots x_{i-1}\, x_i'\, x_{i+1}, \dots x_k)$

f is ESSENTIALLY UNARY if $\exists i \in \{1 \dots k\}$ and unary f_0 s.t.

$$f(x_1 \dots x_k) \approx f_0(x_i)$$

f is NOT essentially unary $\Rightarrow$ f is ESSENTIAL

EQUIVALENTS TO ESSENTIALLY UNARY Let $f \in \mathcal{O}_B$. tfae

① f is essentially unary

② f preserves $P_B^3 := \{(a,b,c) \in B^3 \mid a=b \vee b=c\}$

③ f preserves $P_B^4 := \{(a,b,c,d) \in B^4 \mid a=b \vee c=d\}$

④ f depends on at most one argument.

proof:

① ⇒ ② Let $\bar{a}_1 \ldots \bar{a}_n \in P_B^3$.

$$f(\bar{a}_1, \ldots, \bar{a}_k) \overset{\text{WLOG}}{=} f_0(\bar{a}) = \begin{pmatrix} f_0(a_1) \\ f_0(a_2) \\ f_0(a_3) \end{pmatrix} \in P_B^3$$

for $a_1 = a_2$, $f(a_1) = f(a_2) \Rightarrow \in P_B^3$

similarly for $a_2 = a_3$

✓

② ⇒ ③: Assume by contrapositive f does not preserve P_B^4. Permuting arguments

$\exists\, a^1 \ldots a^k \in P_B^4$ with $\underline{f(a^1, \ldots, a^k) \notin P_B^4}$, $a_1 \ldots a_\ell$ agree on first two coordinates

$a_{\ell+1} \ldots a_k$ agree on last two coord.

Let $c = (a_1^1 \ldots a_1^\ell\; a_4^{\ell+1} \ldots a_4^k)$

$f(\underbrace{a_1^1 \ldots a_1^k}_{d}) \neq f(a_2^1 \ldots a_2^k)$ so, $f(c)$ differs from one of them, i.e. $f(c) \neq f(d)$

$f(\underbrace{a_3^1 \ldots a_3^k}_{e}) \neq f(a_4^1 \ldots a_4^k)$ so $f(c) \neq f(e)$.

Now, $(d^i, c^i, e^i) \in P_B^3$ by constr. But $(f(d), f(c), f(e)) \notin P_B^3$ as $f(c) \neq f(d)$
$\neq f(e)$

EQUIVALENTS TO ESSENTIALLY UNARY

Let $f \in \mathcal{O}_B$. TFAE

1. f is essentially unary
2. f preserves $P_B^3 := \{ (a,b,c) \in B^3 \mid a=b \vee b=c \}$
3. f preserves $P_B^4 := \{ (a,b,c,d) \in B^4 \mid a=b \vee c=d \}$
4. f depends on at most one argument.

<u>proof</u> (cont).

<u>③ ⇒ ④</u> By CONTRAPOSITIVE suppose f depends on ith and jth arguments

$\exists\, a_1\, b_1\, a_2\, b_2 \in B^k$ s.t. $a_1\ b_1$ and $a_2\ b_2$ differ at i and j respectively

$f(a_1) \neq f(b_1)$ $\quad f(a_2) \neq f(b_2)$

Then, $(a_1(l), b_1(l), a_2(l), b_2(l)) \in P_B^4$ $\forall\, l \leq k$ but

$(f(a_1), f(b_1), f(a_2), f(b_2)) \notin P_B^4$ so f does not preserve P_B^4.

<u>④ ⇒ ①</u>: suppose WLOG f only depends on 1st argument.

Let $g(x_1 \dots x_i)$ be minimal in i s.t. $f(x_1 \dots x_n) \approx g(x_1 \dots x_i)$

$i=1 \Rightarrow f$ is unary.

otw, since f does not depend on ith argument, neither does g and so $\exists\, g'(x_1 \dots x_{i-1})$ s.t. $f(x_1 \dots x_n) \approx g(x_1 \dots x_i) \approx g'(x_1 \dots x_{i-1})$ contradicting minimality. ∎

EXAMPLE: All polymorphisms of

$$\mathbb{B} := (\mathbb{Z}; \underline{0}, \{(x,y) \mid x = y+1\}, \underbrace{\{(u,v,x,y) \mid u = v \vee x = y\}}_{P^4_B})$$

are projections.

all polyms are essentially unary.

$$f\begin{pmatrix} y+1 \\ y \end{pmatrix} = \begin{pmatrix} z+1 \\ z \end{pmatrix} \quad \text{so} \quad f(y+1) = f(y)+1$$

$f(0) = 0 \Rightarrow f(1) = 1 \quad \ldots \quad$ so $\quad f = \mathrm{Id}_{\mathbb{Z}}$

all polys are projections.

$$\mathrm{Inv}\,\mathrm{Pol}(\mathbb{B}) \supsetneq \langle \mathbb{B} \rangle_{pp}$$

EQUIVALENTS TO ALL POLYS BEING ESSENTIALLY UNARY

Let B be countable ω-categorical. TFAE:

① All relations with an $\exists^+$ def in B have a pp-def

② P^3_B is pp-def in B

③ All polys of B are essentially unary

Proof:

① ⇒ ② because $P^3_B := (x=y) \vee (y=z)$ is $\exists^+$

② ⇒ ③: since polys preserve pp-formulas + previous lemma.

③ ⇒ ①: Because unary polys preserve $\exists^+$-formulas.

+ $\mathrm{Inv\,Pol}(B) = \langle B \rangle_{pp}$ (so $\exists^+$-formulas must be pp).

§6.1.5 ELEMENTARY CLONES

$f \in Pol(B)$ is ELEMENTARY if it preserves fo-formulas

If every $f \in Pol(B)$ is elementary, we say $Pol(B)$ is ELEMENTARY.

EQUIVALENTS TO POL(B) elementary B countable ω-categorical. tfae

1. Every relation with a fo def also has a pp-def
2. B is a model complete core + P^3_B is pp-def in B
3. Pol(B) is locally generated by many operations invertible in Pol(B)
4. Pol(B) is elementary

Proof:

① ⇒ ② B is a mc core iff every fo formula is ≡ to an $\exists^+$ one

So B is a mc core. P^3_B is fo def, so it has a pp-def.

② ⇒ ③: B is a core iff $Aut(B) = End(B)$ ⇒ $\langle Aut(B) \rangle = Pol(B)$

$P^3_B \in \langle B \rangle_{pp}$ ⇒ $\langle End(B) \rangle = Pol(B)$

so $\overline{\langle Aut(B) \rangle} = Pol(B)$

by Prop 6.1.5: smallest locally closed clone $\supseteq S$ $= \overline{\langle S \rangle}$

③ ⇒ ④: Automorphisms preserve f.o. formulas.

$Pol(B) = \overline{\langle Aut(B) \rangle} \overset{\text{Prop 6.1.5}}{=} Pol(\underbrace{Inv(Aut(B))}_{\text{fo formulas}})$ ⇒ all polys preserve fo formulas.

④ ⇒ ①: by $Inv(Pol(B)) = \langle B \rangle_{pp}$ ⇒ fo formulas are ≡ to pp-formulas. ☒

COROLLARY 6.1.21 B ω-CAT + ctble + $|B| > 1$.
Pol(B) is elementary ⇒ B pp-interprets all finite structures

__Proof__: Since all finite structures have a fo def in B + Previous Lemma.

Lemma 6.1.22 B ω-categorical with all polys essentially unary. Then the mc core of B has an elementary poly clone.

__Proof__:
Since all polys are essentially unary, P^3_B has a pp-def in B (given by ϕ)
__WNTS__: ϕ is a pp-def of P^3_B in $C = \text{core}(B)$. This is routine ▨

COROLLARY 6.1.23 B ω-CAT + ctble + no constant endomorphism + all polys are essentially unary.
Then, B has a finite signature reduct with NP-hard CSP.

__Proof__: No constant endomorphism ⇒ $|B| > 1$. Let $C = \text{core}(B)$, and $|C| > 1$ by no const end.
Pol(C) is ELEMENTARY by Lemma 6.1.22.
So C pp-interprets all finite structures. So

$K_3 \in \underbrace{I(C)}_{C \overset{\text{hom}}{\hookrightarrow} B} \subseteq I(H(B)) \underbrace{\subseteq}_{\text{by WONDERLAND}} HI(B)$ ⇒ B has a finite sign. reduct with NP-hard CSP. ▨