§ 6.1.8 (cont) MINIMAL CLONES

Let $\mathcal{C}$ be a closed subclone of $\mathcal{O}$. We say that f is MINIMAL ABOVE $\mathcal{C}$ if f is of minimal arity in the set

$$\{ g \in \mathcal{O} \setminus \mathcal{C} \mid \forall h \in \mathcal{O} \setminus \mathcal{C} \, (g \in \overline{\langle \mathcal{C} \cup \{h\} \rangle} \Rightarrow h \in \overline{\langle \mathcal{C} \cup \{g\} \rangle}) \}$$

The closed clone $\mathcal{D} \supsetneq \mathcal{C}$ is MINIMAL ABOVE $\mathcal{C}$ if $\nexists \mathcal{E}$ closed s.t. $\mathcal{C} \subsetneq \mathcal{E} \subsetneq \mathcal{D}$.

- minimal clones are generated by minimal operations (and vice versa)
- minimal clones EXIST IN AN OLIGOMORPHIC, finite language, context.

if $\underset{\text{closed}}{\mathcal{C}} \supsetneq \mathrm{Pol}(\underset{\omega\text{-cat}}{\mathbb{B}})$ $\quad \exists \, \mathcal{C} \supseteq \mathcal{D} \supsetneq \mathrm{Pol}(\mathbb{B})$ minimal above $\mathrm{Pol}(\mathbb{B})$.

A k-ary operation f is

- symmetric if f is binary & $f(x,y) \approx f(y,x)$
- quasi NEAR-UNANIMITY if $k \geq 3$ and
  $f(x,\dots,x,y) \approx f(x,\dots,y,x) \approx \dots \approx f(y,x,\dots x) \approx f(x,\dots,x)$
- quasi MAJORITY if $K=3$ and f is quasi-NU
  (so $f(x,x,y) \approx f(x,y,x) \approx f(y,x,x) \approx f(x,x,x)$)
- quasi MINORITY if $k=3$ and
  $f(y,y,x) \approx f(y,x,y) \approx f(x,y,y) \approx f(x,x,x)$
- quasi Mal'cev if $k=3$
  $f(x,y,y) \approx f(y,y,x) \approx f(x,x,x)$
- quasi SEMIPROJECTION if $\exists\, i \in \{1,\dots,k\}$ and unary g s.t.
  whenever $|\{a_1,\dots,a_k\}| < k$
  $$f(a_1,\dots,a_k) = g(a_i)$$

we write $\hat{f}(x)$ for $f(x,\dots,x)$

f is a weak semiprojection if $\forall\, i \neq j \in \{1, \ldots, n\} = [n]$ $\exists\, s(i,j)$ and a unary non-constant operation $g_{i,j}$ s.t.

for all $(a_1, \ldots, a_n)$ s.t. $a_i = a_j$,

$$f(a_1, \ldots, a_n) = g_{i,j}(a_{s(i,j)})$$

Some notation:

Let $S \subseteq [n]$ with $|S| > 2$.

Then, $\exists\, k \in [n]$ s.t. $\forall\, (a_1, \ldots, a_n)$ s.t. $a_i = a_j$ for all $i, j \in S$,

$$f(a_1, \ldots, a_n) = g_{i,j}(a_k).$$

We define

$$E(S) := \begin{cases} S & \text{if } k \in S; \\ \{k\} & \text{otherwise.} \end{cases}$$

(*) $S \subseteq T \Rightarrow E(S) \subseteq E(T)$

- If $\exists\, k \in [n]$ s.t. $k \in E(S)$ for all $S \subseteq [n]$ with $|S| > 2$, f is a quasi-semiprojection.

WEAK $\Rightarrow$ QUASI Let f be a weak semiprojection of arity $n \geq 4$. Then, f is a quasi-semiprojection.

Proof:

CLAIM 1: For $I, J \subseteq [n]$ s.t. $|I| = |J| = 2$, $I \cap J = \emptyset$, $E(I) \cap E(J) \neq \emptyset$.

Proof: WLOG let $I = \{1,2\}$, $J = \{3,4\}$.

- $E([4]) = \{\ell\} \overset{(*)}{\Rightarrow} E(I) = E(J) = \{\ell\}$ ✓
- $E(I) = \{i\} \subseteq J$
  - $E(J) = J$. Then $E(I) \cap E(J) \neq \emptyset$ ✓
  - $E(J) = \{j\} \subseteq I$. Then

$g_{12}(y_1) = g_{34}(x) = g_{12}(y_2)$

$$g_{12}(y) \approx f(x,x,y,y,x_5,\dots) \approx g_{34}(x) \Rightarrow g_{12} \text{ and } g_{34} \text{ are constant } \#$$

- $E(I) = I$, $E(J) = J$. Then, again

$$g_{34}(y) = f(x,x,y,y,x_5 \dots x_n) = g_{12}(x). \ \#$$

So $E(I) \cap E(J) \neq \emptyset$ ∎

Let $i \in E(\{1,2\}) \cap E(\{3,4\})$.

CLAIM 2: For $T \subseteq \{1,\dots,n\}$ $|T|>2$, $i \in E(T)$.

proof (a) If $T \subseteq [n]\setminus\{i\}$. $E(T) \subseteq E([n]\setminus\{i\}) = \{i\}$ (Δ)

So $E(T)=\{i\}$.

either $\{1,2\}$ or $\{3,4\}$ is $\subseteq [n]\setminus\{i\}$
so $i \in E([n]\setminus\{i\})$. So
$E([n]\setminus\{i\}) \neq [n]\setminus\{i\}$ and so is $\{i\}$.

(b) suppose $i,j \in T$ By CLAIM 1 $\exists k$ s.t

$k \in E(\{i,j\}) \cap E([n]\setminus\{i,j\}) \neq \emptyset$.

- $k \notin \{i,j\} \Rightarrow E([n]\setminus\{i,j\}) = [n]\setminus\{i,j\}$ ※ since $E([n]\setminus\{i\})=\{i\}$ by (Δ)

So $E(\{i,j\}) = \{i,j\}$.

So $E(T)=T$ for $|T|>2$ and $i \in T$.

So in both cases $i \in E(T)$. ☒

So $i \in E(T)$ for every $T \subseteq \{1,\dots,n\}$ with at least two elements.

So f is a quasi-semiprojection. $\square$

FIVE TYPES THEOREM (generalising Rosenberg's)

Let $\mathcal{C}$ be an essentially unary closed clone.
Let f be a minimal operation above $\mathcal{C}$. Then f is, up to permuting variables, one of the following types:

1. unary;
2. binary;
3. a ternary quasi-majority;
4. quasi-Malcev; $f(x,x,y)=f(y,x,x)=f(y,y,y)$
5. a k-ary quasi-semiprojection for $k \geq 3$.

Proof: Let f be ternary.

$f_1(x,y) := f(y,x,x)$, $f_2(x,y) := f(x,y,x)$, $f_3(x,y) := f(x,x,y)$.

By minimality $f_i(x,y) = \hat{f}(x)$ or $= \hat{f}(y)$ where $\hat{f}(x) = f(x,x,x)$.
because each must be in $\mathcal{C}$ and so essentially unary.

We can then check each case.

f_1	f_2	f_3	type
$\hat{f}(x)$	$\hat{f}(x)$	$\hat{f}(x)$	quasi majority
$\hat{f}(x)$	$\hat{f}(x)$	$\hat{f}(y)$	quasi semiprojection
$\hat{f}(x)$	$\hat{f}(y)$	$\hat{f}(x)$	quasi semiprojection
$\hat{f}(x)$	$\hat{f}(y)$	$\hat{f}(y)$	quasi Maltsev
$\hat{f}(y)$	$\hat{f}(x)$	$\hat{f}(x)$	quasi semiprojection
$\hat{f}(y)$	$\hat{f}(x)$	$\hat{f}(y)$	quasi Maltsev
$\hat{f}(y)$	$\hat{f}(y)$	$\hat{f}(x)$	quasi Maltsev
$\hat{f}(y)$	$\hat{f}(y)$	$\hat{f}(y)$	quasi Maltsev

Also note that for f k-ary for $k > 3$, f is a weak semiprojection.

Hence, by PREVIOUS LEMMA, f is a quasi-semiprojection. ▨

IMPROVEMENTS:

QUASI MALCEV CANNOT HAPPEN Let $G \curvearrowright B$ be a non-trivial group acting faithfully, though not freely, on a set B. Then, there are no quasi-Malcev operations minimal above $\overline{\langle G \rangle}$.

Proof:

Let $\alpha \in G \setminus \{1\}$ be s.t. $\alpha a = a$ for some $a \in B$.

Let $b \neq c \in B$ be s.t. $\alpha b = c$.

$h(x,y) := M(x, \alpha x, y)$ must depend entirely on x or on y by minimality.

- $h(x,y)$ depends on x:

  $h(a,a) = M(a, \alpha a, a)$ $\qquad h(a,b)$

  $$M(a,a,a) = M(a,a,b) = M(b,b,b) \quad ⨳$$ contradicts injectivity of $h(x)$.

- $h(x,y)$ depends on y and is given by g:

  $$g(b) = M(a,a,b) = M(b,b,b) = M(b,c,c) = g(c) \quad ⨳$$

So we cannot have a quasi-Malcev minimal above $\overline{\langle G \rangle}$. □

REMEMBER: $G \curvearrowright B$ freely if $\alpha a = a \Rightarrow \alpha = 1$.

If $G \curvearrowright B$ is oligomorphic the action is not free.

FROM BODIRSKY-CHEN (2007)

THEOREM 6.1.45. *Let $\mathscr{G}$ be an* ~~*oligomorphic permutation*~~ weaker *group on a countably infinite set B with r orbitals and s orbits, and let f be minimal above $\overline{\langle \mathscr{G} \rangle}$. Then f is of one of the following types:*

(1) A unary operation.

(2) A binary operation.

~~*(3) A ternary quasi majority operation.*~~

(4) A k-ary quasi semiprojection, for $3 \leq k \leq 2r - s$.

In ongoing work, we can improve this to:

Theorem 2.7 (Three types theorem). *Let $G \curvearrowright B$ be such that G is not a Boolean group acting freely on B. Let s be the (possibly infinite) number of orbits of G on B. Let f be a minimal operation above $\overline{\langle G \rangle}$. Then, f is of one of the following types:*

1. *f is unary;*
2. *f is binary;*
3. *f is a k-ary orbit-semiprojection for $3 \leq k \leq s$.*

f is a k-ary orbit-semiprojection if there is $i \in \{1, \dots, k\}$ and $g \in \overline{\langle G \rangle}$ s.t. for all $(a_1, \dots, a_k)$ with at least two entries in the same G-orbit

$$f(a_1, \dots, a_k) = g(a_i).$$

WE CAN SOLVE A QUESTION AT THE END of the book:

(24) Does every countably infinite ω-categorical core with an essential polymorphism also have a binary essential polymorphism?

Corollary 3.8. *Let B be an ω-categorical countable model complete core such that $\mathrm{Aut}(B)$ has ≤ 2 orbits. Then, if $\mathrm{Pol}(B)$ has an essential polymorphism, it also has a binary essential polymorphism. Moreover, this binary polymorphism is minimal above $\overline{\langle \mathrm{Aut}(B) \rangle}$.*

Proof. Since B is an ω-categorical model complete core, $\mathrm{Pol}(B) \cap \mathcal{O}^{(1)} = \overline{\mathrm{Aut}(B)}$ (Definition 3.4). Since $\mathrm{Pol}(B)$ contains an essential polymorphism, $\mathrm{Pol}(B) \supsetneq \overline{\langle \mathrm{Aut}(B) \rangle}$. Moreover, by Fact 3.2, $\mathrm{Pol}(B)$ contains a closed subclone $\mathcal{C}$ which is minimal above $\langle \overline{\mathrm{Aut}(B)} \rangle$. Let f be the minimal function such that $\overline{\langle \mathrm{Aut}(B) \cup \{f\} \rangle} = \mathcal{C}$. Since $\mathrm{Pol}(B) \cap \mathcal{O}^{(1)} = \overline{\mathrm{Aut}(B)}$, f is not unary. By the three-types theorem (Theorem 2.7), since $s \leq 2$, f has to be binary. Clearly, f is also essential. $\square$

Corollary 3.9. *Let B be a countable ω-categorical structure such that $\mathrm{Aut}(B)$ has $s \geq 3$ orbits on B. Then, for each $3 \leq k \leq s$, B has a first order reduct which is a model complete core and such that it has a k-ary essential polymorphism and no essential polymorphisms of lower arity.*