Thm 6.2.1 (Post) Let $\mathcal{C}$ be an essentially unary operation clone with domain $\{0,1\}$. Then every minimal operation above $\mathcal{C}$ is among one of the following:

- a unary operation
- the binary $(x,y) \mapsto \min(x,y)$
- the binary $(x,y) \mapsto \max(x,y)$
- the minority operation
- the majority operation

Corollary 6.2.2 All polymorphisms of $(\{0,1\}; 1IN3)$ are projections.

Proof: $1IN3$ is not preserved by neither min, max, minority, or majority. The only unary polymorphism is identity.
$\hookrightarrow \begin{pmatrix}0\\0\\1\end{pmatrix} \mapsto \begin{pmatrix}1\\1\\0\end{pmatrix} \notin 1IN3$ $\square$

Def 6.2.4 A propositional formula φ in CNF is called *reduced* if whenever we remove a literal from a clause in φ, the resulting formula is not equivalent to φ.

Normal forms for Boolean relations

Preserved by...	definition by...
minority	conjunction of lin. eq. mod 2
min	conjunction of Horn clauses, i.e. $\bigwedge_i (A^i \vee \neg B^i_1 \vee \dots \vee \neg B^i_k)$
majority	conjunction of clauses of size two, i.e. $\bigwedge_i (A_i \vee B_i)$
max	conjunction of dual-Horn clauses, i.e. $\bigwedge_i (\neg A^i \vee B^i_1 \vee \dots \vee B^i_k)$

Theorem 6.2.7 Let $\mathfrak{B}$ be a structure with finite signature ...

Theorem 6.1.42 Let $\mathcal{C}$ be an essentially unary clone and let f be a minimal operation above $\mathcal{C}$. Then f is one of the following types:

- a unary operation
- a binary essential operation s.t. $\hat{f} \in \mathcal{C}$
- a quasi Malcev
- a quasi majority
- a k-ary quasi semiprojection for $k \geq 3$

Proof for minority-NF

Let R n-ary, $R \subseteq \{0,1\}^n$

R affine iff solution space of system of lin. eq. mod 2.

affine spaces are precisely those closed under affine comb., so $\alpha_1 x_1 + \dots + \alpha_k x_k$ s.t. $\sum_i \alpha_i = 1$.

If R affine $\stackrel{\text{}}{=}$ sol. sp. of lin. eq.: R preserved by $(x_1,x_2,x_3) \mapsto x_1+x_2+x_3$

This is minority mod 2.

If R preserved by minority: then we can write $x_1 + \dots + x_k$, k odd
$\text{minority}(x_1, x_2, \text{minority}(x_3, x_4, \text{minority}(\dots x_{k-1}, x_k))\dots)$
this is contained in R. $\leadsto$ Def by lin. eq.

Proof for min-NF: (and max)

R can be defined as Horn, i.e. every clause is a disjunction with at most one positive literal.

$a, b \in R$ ... $R = \bigwedge(x \vee \neg y_1 \vee \dots$

Proof of Thm 6.2.1:

Let f minimal above $\mathcal{C}$, arity ≥ 2.
$\leadsto \hat{f} \in \mathcal{C}$ cannot be constant.
$\hat{f}$ is either identity $\leadsto f$ is idempotent
$\hat{f}$ is $\neg$ $\leadsto \neg f$ is idempotent
$\neg f \in \mathcal{C}$.

Wlog f idempotent.

There are only 4 binary idempotent operations on $\{0,1\}$, two are projections – projections are not minimal
– the only other two options are min, max

Next: Semiprojections of arity ≥ 3 on $\{0,1\}$ are projections.

So: either Malcev or Majority.

If $f(x,y,x) = y \leadsto f$ minority

If $f(x,y,x) = x$

$g(x,y,z) := f(x, f(x,y,z), z)$

fulfills
$$\left.\begin{aligned} g(xxz) &= x \\ g(xyy) &= y \\ g(xyx) &= x \end{aligned}\right\} g \text{ majority in Clone } \lightning\ f \text{ minimal.}$$
$\square$

Theorem 6.2.7 (Schaefer) Let $\mathfrak{B}$ be a structure with finite signature over $\{0,1\}$. Then either $(\{0,1\}; 1IN3)$ has a pp-definition in $\mathfrak{B}$, and CSP($\mathfrak{B}$) is NP-complete, or:

(1) $\mathfrak{B}$ is preserved by a constant operation.
(2) $\mathfrak{B}$ is preserved by min.
(3) $\mathfrak{B}$ is preserved by max.
(4) $\mathfrak{B}$ is preserved by majority.
(5) $\mathfrak{B}$ is preserved by minority.

In case (1-5) the problem CSP($\mathfrak{B}$) is in P.

Proposition 6.2.8 Let $\mathfrak{B}$ be a structure over a two-element universe. Then the following are equivalent:

(1) The relation NAE has a pp-def. in $\mathfrak{B}$.
(2) $\mathfrak{B}$ is preserved neither by min, max, minority, majority, nor the constant operation.
(3) Either the polym. clone of $\mathfrak{B}$ contains only projections, or it is generated by $x \mapsto \neg x$.
(4) Every f.-o. formula is equivalent over $\mathfrak{B}$ to a pp-formula.

Proof of 6.2.8

(1)⇒(2): NAE is not preserved by min, max, minority, majority, constant.

(2)⇒(3): unary ops which remain (only candidates for min op) are id or ¬id

(3)⇒(4): 6.1.20 (there (3)⇒(1))

(4)⇒(1):

$NAE(x,y,z) :\Leftrightarrow (x \neq y \vee y \neq z)$ □

... is a disjunction with at most one positive literal.

$a, b \in R$ ... $R = \bigwedge(x \vee \neg y_1 \vee \dots \vee \neg y_k)$

If any of y_i^a or y_i^b are 0 then also $\min(y_i^a, y_i^b) = 0$ (✓)

So consider $y_i^a, y_i^b = 1$

x^a, x^b need to be 1

min is 1 ⇝ $\in R$.

Assume R preserved by min.
φ reduced CNF defining R.

Suppose φ contains clause C with more than one positive literal.

$C = (A \vee B \vee \neg C \vee \dots)$

Reduced: There are s_1: s.t. $s_1(A) = 1$, $s_1(y) = 0$

There are s_2: s.t. $s_2(B) = 1$, $s_2(y) = 0$

Then $\min(s_1, s_2) = (0\ 0 \dots)$ does not satisfy C, and is not in R, so R not preserved by min ↯ □

The max-operation has the same in dual for dual-Horn □

Proof for majority-NF:

R has a bijunctive definition. $\Rightarrow$ (Bake Pixly) preserved by majority.

R preserved by majority.

φ reduced CNF defining R.

$\square$

Theorem: (Baker-Pixley) Let A finite, with a $(d+1)$-ary near-unanimity operation. Then any function $f: A^n \longrightarrow A$ is a term function of A iff f preserves all subuniverses of A^d.

$\boxed{\text{Majority: } d=2}$

R preserved by majority.

φ reduced CNF defining R.

Assume clause C with three literals ℓ_1, ℓ_2, ℓ_3

Reduced $\rightsquigarrow \exists s_1, s_2, s_3$ s.t. s_i only ℓ_i eval. to 1

majority$(s_1(x), s_2(x), s_3(x)) \rightsquigarrow 0$ everywhere

So majority does not preserve R $\lightning$ $\square$

Proof of 6.2.7 (Schaefer)

(1) $Pol(\mathfrak{B}) \ni$ constant

$\Rightarrow$ one element core

1.1.12: $CSP(\mathfrak{B}) \in P$. (✓)

Assume: We don't have constant.

If NAE is pp-definable: by 1.2.8 $CSP(\mathfrak{B})$ is NP-hard.

Assume now: NAE is not pp-definable

by 6.1.2 there is $f \in Pol(\mathfrak{B})$ which does not preserve NAE.

If $\hat{f} = id \Rightarrow f$ idemp.

$\hat{f} = \neg id \Rightarrow \neg f$ idemp.

So wlog assume f idempotent.

f generates minimal operation $g \in Pol(\mathfrak{B})$

g arity ≥ 2.

By 6.2.1: $g = \begin{cases} \text{min} \\ \text{max} \\ \text{minority} \\ \text{majority} \end{cases}$

Case 1) $g = \min$, or $g = \max$

$\hookrightarrow$ Know from Literature that Horn, dual-Horn are in P.

Case 2) $g =$ majority:

2SAT is in P (Literature).

Case 3) $g =$ minority:

System of lin. eq. mod 2

$\rightsquigarrow$ Gauß in P.

$\square$