CSP Reading Group: Section 6.3

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OBSERVATION: TFAE for $f: A^n \to A$ and $R \subseteq A^m$

(1) $f \in Pol(A; R)$

(2) f preserves R

(3) f is a homomorphism $(A; R)^n \to (A; R)$

(4) R is a subalgebra of $(A; f)^m$

DEF. congruence of an algebra $\underline{A}$ = equivalence relation preserved by all operations of $\underline{A}$ ($\Leftrightarrow$ subalgebra of $\underline{A}^2$)

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Section 6.1.2 $\Rightarrow$ $\mathbb{A}$ count. ω-categorical, $\underline{A}$ its polymorphism algebra ... congruences of $\underline{A}$ $\equiv$ pp-definable equivalences over $\mathbb{A}$

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PROPOSITION: C is a congruence of $\underline{A}$ $\Leftrightarrow$ C is the kernel of some homomorphism $h: \underline{A} \to \underline{B}$, i.e.,

$$C = \{(a_1, a_2) \in A^2 \mid h(a_1) = h(a_2)\}$$

DEF: C a congruence of τ-algebra $\underline{A}$ ⇝ quotient algebra $\underline{A}/C$ denotes τ-algebra with the domain A/C where

$$f^{A/C}(a_1/C, \dots, a_n/C) = f^{A}(a_1, \dots, a_n)/C$$

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LEMMA: $\underline{A}, \underline{B}$ τ-algebras, $h: \underline{A} \to \underline{B}$ homomorphism

For every $\underline{A'} \leq \underline{A}$, $h(\underline{A'}) \leq \underline{B}$.

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Proof: $\forall a_1, \dots, a_n \in A'$: $f^{\underline{B}}(h(a_1), \dots, h(a_n)) = h(f^{\underline{A}}(a_1, \dots, a_n)) \in h(A')$

$\Rightarrow h(\underline{A'}) \leq \underline{B}$

$\forall a_1, \dots, a_n \in h^{-1}(\underline{B'})$: $h(f^{\underline{A}}(a_1, \dots, a_n)) = f^{\underline{B}}(h(a_1), \dots h(a_n)) \in \underline{B'}$

$\Rightarrow f^{\underline{A}}(a_1, \dots, a_n) \in h^{-1}(\underline{B'}) \Rightarrow h^{-1}(\underline{B'}) \leq \underline{A}$.

$\mathcal{K}$ ... class of algebras of the same signature

- $H(\mathcal{K})$ = class of all homomorphic images of algebras from $\mathcal{K}$
- $S(\mathcal{K})$ = — " — subalgebras — " —
- $P(\mathcal{K})$ = — " — products — " —
- $P^{fin}(\mathcal{K})$ = — " — finite products — " —
- $Exp(\mathcal{K})$ = — " — $\underbrace{\text{expansions}}_{\text{adding operations}}$ — " —

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DEF.: $\mathcal{K}$ is

- a pseudo-variety if $H(\mathcal{K}) = S(\mathcal{K}) = P^{fin}(\mathcal{K}) = \mathcal{K}$,
- a variety if $H(\mathcal{K}) = S(\mathcal{K}) = P(\mathcal{K}) = \mathcal{K}$.

the smallest (pseudo-) variety containing $\mathcal{K}$ = (pseudo-) variety generated by $\mathcal{K}$

LEMMA:

(1) The pseudo-variety generated by $\mathcal{K}$ is $HSP^{fin}(\mathcal{K})$.

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Proof: We show (2), (1) is analogous.

Enough to show $\underline{SH(\mathcal{K}) \subseteq HS(\mathcal{K})}$, $\underline{PS(\mathcal{K}) \subseteq SP(\mathcal{K})}$ and $\underline{PH(\mathcal{K}) \subseteq HP(\mathcal{K})}$. Then

$HHSP(\mathcal{K}) = HSP(\mathcal{K})$, $SHSP(\mathcal{K}) \subseteq HSSP(\mathcal{K}) = HSP(\mathcal{K})$,

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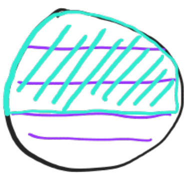
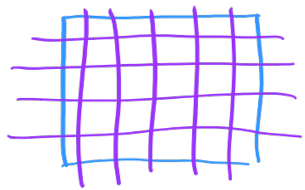
Proof by pictures (courtesy of Libor Barto):

$SH(\mathcal{K}) \subseteq HS(\mathcal{K})$ $\qquad$ $PS(\mathcal{K}) \subseteq SP(\mathcal{K})$ $\qquad$ $PH(\mathcal{K}) \subseteq HP(\mathcal{K})$





PROPOSITION: If $\underline{\mathbb{C}}$ is a polymorphism algebra of a structure $\mathbb{C}$ and $\mathbb{B} \in \mathcal{I}(\mathbb{C})$, then $\mathrm{Exp}\,\mathrm{HSP}^{fin}(\underline{\mathbb{C}})$ contains a polymorphism algebra of $\mathbb{B}$.

PROPOSITION: If $\underline{C}$ is a polymorphism algebra of a structure $\mathbb{C}$ and $\mathbb{B} \in I(\mathbb{C})$, then $\operatorname{Exp} HSP^{fin}(\underline{C})$ contains a polymorphism algebra of $\mathbb{B}$.

Proof:

Suppose that $\mathbb{B}$ has a d-dimensional pp-interpretation I in $\mathbb{C}$.

$I^{-1}(B)$ pp-definable in $\mathbb{C}$ => preserved by all operations => induces a subalgebra $\underline{D} \leq \underline{C}^d$

$K :=$ kernel of I $\qquad K = I^{-1}(=_B)$ is pp-definable in $\mathbb{C}$ => => preserved by all operations => K is a congruence of $\underline{D}$

$\rightsquigarrow$ $\underline{B} := \underline{D}/K$, $I : \underline{D} \to \underline{D}/K$ is surjective

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$Clo(\underline{B}) \leq Pol(\mathbb{B})$ since all the relations of $\mathbb{B}$ are pp-definable as relations over C^d and hence subalgebras of the corresponding power of $\underline{B}$

THEOREM: $\mathbb{C}$ countable ω-categorical, $\underline{C}$ polym. algebra of $\mathbb{C}$

(1) $\mathbb{B} \in I_{full}(\mathbb{C}) \iff \exists\, \underline{B} \in HSP^{fin}(\underline{C})$ s.t.

$$\overline{Clo(\underline{B})} = Pol(\mathbb{B}).$$

(2) $\mathbb{B} \in \underbrace{Red(\mathbb{C})}_{\text{pp-reducts}} \iff \exists\, \underline{B} \in Exp(\underline{C})$ s.t.

$$Clo(\underline{B}) = Pol(\mathbb{B})$$

(3) $\mathbb{B} \in I(\mathbb{C}) \iff \exists\, \underline{B} \in Exp\,HSP^{fin}(\underline{C})$ s.t.

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$Clo(\underline{B}) = Pol(\mathbb{B})$

Proof:

(2): $\Rightarrow$: $\mathbb{B}$ ω-categorical + $\langle \mathbb{B} \rangle_{pp} = Inv\,Pol(\mathbb{B})$

$\Leftarrow$: $Clo(\underline{B})$ oligomorphic + $\langle \mathbb{B} \rangle_{pp} = Inv\,Pol(\mathbb{B})$

(1) + (2) $\Rightarrow$ (3): Recall $\mathbb{B} \in I(\mathbb{C}) \iff \mathbb{B} \in Red\,I_{full}(\mathbb{C})$

The closure is not needed, because in Exp one can add those operations.

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Proof:

$\Rightarrow$: The same proof as for Proposition before, but we want $\overline{Clo(\underline{B})} = Pol(\mathbb{B})$ instead of $Clo(\underline{B}) \subseteq Pol(\mathbb{B})$ (no Exp allowed).

Lemma 4.7.3 $\Rightarrow$ $\mathbb{B}$ is ω-categorical as well.

So $\underline{B} = \underline{D}/K$ where $\underline{D} \leq \underline{C}^d$, $K = I^{-1}(=_B)$.

Enough: R pp-definable in $\mathbb{B}$ $\iff$ preserved by all operations of $\underline{B}$

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$f \in \tau$: $f^{\underline{B}}$ preserves R $\iff f^{\underline{C}}$ preserves $I^{-1}(\underline{R}) \iff$

$\iff I^{-1}(R)$ pp-definable in $\mathbb{C}$ $\overset{\text{full interpretation}}{\iff}$ R pp-def. in $\mathbb{B}$

(1) $\mathbb{B} \in I_{full}(\mathbb{C}) \iff \exists\, \underline{B} \in HSP^{fin}(\underline{C})$ s.t.

$\overline{Clo(\underline{B})} = Pol(\mathbb{B})$.

Proof:

$\Leftarrow$: $\exists\, d \in \mathbb{N}$ $\quad \underline{D} \leq \underline{C}^d$ $\quad h: \underline{D} \to \underline{B}$ surjective homomorphism

Claim: h is a full pp-interpretation of $\mathbb{B}$ in $\mathbb{C}$

- all operations of $\underline{C}$ preserve $D \subseteq C^d$
  $\Rightarrow D$ is pp-definable in $\mathbb{C}$ $\rightsquigarrow$ domain formula
- $K :=$ kernel of h is a congruence $\to$ pp-definable in $\mathbb{C}$
  $\rightsquigarrow =_B$-formula

- $R \subseteq B^k$ relation of $\mathbb{B}$

  $\tau :=$ signature of $\underline{\mathbb{C}}$, $f \in \tau$

  $f^{\underline{\mathbb{B}}}$ preserves R $\Rightarrow$ $f^{\underline{\mathbb{C}}}$ preserves $h^{-1}(R)$

  $\rightsquigarrow$ $\mathrm{Pol}(\mathbb{C})$ preserves $h^{-1}(R) \Rightarrow h^{-1}(R)$ is pp-definable in $\mathbb{C}$

  $\rightsquigarrow$ interpreting formula for R

$\Rightarrow$ h is a pp-interpretation of $\mathbb{B}$ in $\mathbb{C}$

$\xRightarrow{\text{LEMMA 4.7.3}}$ $\mathbb{B}$ is ω-categorical

- $R \subseteq B^k$ relation of $\mathbb{B}$
  $\tau :=$ signature of $\underline{C}$, $f \in \tau$
  $f^{\underline{B}}$ preserves $R \Rightarrow f^{\underline{C}}$ preserves $h^{-1}(R)$
  $\rightsquigarrow$ $\mathrm{Pol}(\mathbb{C})$ preserves $h^{-1}(R) \Rightarrow h^{-1}(R)$ is pp-definable in $\mathbb{C}$
  $\rightsquigarrow$ interpreting formula for R
$\Rightarrow h$ is a pp-interpretation of $\mathbb{B}$ in $\mathbb{C}$
$\xRightarrow{\text{LEMMA 4.7.3}}$ $\mathbb{B}$ is ω-categorical

- $R \subseteq B^k$ s.t. $h^{-1}(R)$ pp-definable in $\mathbb{C}$
  $\Rightarrow h^{-1}(R)$ preserved by $\mathrm{Pol}(\mathbb{C}) = \overline{\mathrm{Clo}(\underline{C})}$
  $\Rightarrow R$ preserved by $\mathrm{Clo}(\underline{B})$, hence also by $\overline{\mathrm{Clo}(\underline{B})} = \mathrm{Pol}(\mathbb{B})$
  $\Rightarrow R$ pp-definable in $\mathbb{B}$

$\Rightarrow h$ is a full pp-interpretation of $\mathbb{B}$ in $\mathbb{C}$

Corollary of the previous proof:

THEOREM: $\mathbb{C}$ countable ω-cat. with polymorphism algebra $\underline{C}$, $\mathbb{B}$ arbitrary structure, $h: C^d \to B$ a partial surjection

TFAE: (1) h is a pp-interpretation of $\mathbb{B}$ in $\mathbb{C}$

(2) h is a surjective homomorphism from $\underline{S} \in S(\underline{C}^d)$ to $\underline{B}$ s.t. $\mathrm{Clo}(\underline{B}) \subseteq \mathrm{Pol}(\mathbb{B})$

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PROPOSITION: $\mathbb{A}, \mathbb{B}$ countable ω-cat. TFAE:

(1) $\mathbb{A}, \mathbb{B}$ have pol. algebras $\underline{A}, \underline{B}$, resp. s.t. $HSP^{fin}(\underline{A}) = HSP^{fin}(\underline{B})$

(2) $\mathbb{A}, \mathbb{B}$ are $\underbrace{\text{pp-bi-interpretable}}$.

$\{\bar{x} \mid I_2 \circ I_1(\bar{x}) = \bar{x}\}$ and $\{\bar{y} \mid I_2 \circ I_1(\bar{y}) = \bar{y}\}$ are pp-definable

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Proof idea:

(1) $\Rightarrow$ (2): Use the theorem above to describe the two pp-interpretations I_1, I_2 and $I_1 \circ I_2$, $I_2 \circ I_1$.

Show pp-homotopy by showing that the respective relations are preserved by $\mathrm{Clo}(\underline{A})$ or $\mathrm{Clo}(\underline{B})$.

THEOREM: $h: D^d \to B$ a partial surjection

TFAE:

(1) h is a pp-interpretation of $\mathbb{B}$ in $\mathbb{C}$

(2) h is a surjective homomorphism from $\underline{S} \in S(\underline{C}^d)$ to $\underline{B}$ s.t. $\mathrm{Clo}(\underline{B}) \subseteq \mathrm{Pol}(\mathbb{B})$

PROPOSITION: $\mathbb{A}, \mathbb{B}$ countable ω-cat. TFAE:

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Proof idea:

(2)$\Rightarrow$(1): $\underline{A}$ pol. algebra of $\mathbb{A}$ $\rightsquigarrow$ $\underline{B} \in HSP^{fin}(\underline{A})$

pol. algebra of $\mathbb{B}$ $\rightsquigarrow$ $\underline{A}' \in HSP^{fin}(\underline{B})$ pol. algebra of $\mathbb{A}$

$HSP^{fin}(\underline{A}') \subseteq HSP^{fin}(\underline{B}) \in HSP^{fin}(\underline{A})$

Show $\underline{A} = \underline{A}'$ by showing $f^{\underline{A}} = f^{\underline{A}'}$ $\forall f \in \tau$ by pp-homotopy.

THEOREM: Let $\mathbb{B}$ be countable ω-cat., $\underline{B}$ pol. algebra of $\mathbb{B}$.

TFAE:

(1) $I(\mathbb{B})$ contains all finite structures;

(2) $I(\mathbb{B})$ contains K_n for some $n \geq 3$;

(3) $I(\mathbb{B})$ contains $(\{0,1\}, NAE)$;

(4) $I(\mathbb{B})$ contains $(\{0,1\}, 1IN3)$;

(5) $I(\mathbb{B})$ contains for $\forall A$ finite some $\mathbb{A}$ with $Pol(\mathbb{A}) = \{\text{projections on } A\} =: Proj_A$

(6) $HSP^{fin}(\underline{B})$ contains $\forall A$ finite some $\underline{A}$ with $Clo(\underline{A}) = Proj_A$.

(7) $HSP^{fin}(\underline{B})$ contains some $\underline{A}$ with $|A| \geq 2$ s.t. $Clo(\underline{A}) = Proj_A$.

(8) $I(\mathbb{B})$ contains $\mathbb{A}$ with $|A| \geq 2$ s.t. $Pol(\mathbb{A}) = Proj_A$.

(9) $I(\mathbb{B})$ contains $\mathbb{A}$ with $|A| \geq 2$ s.t. fo-formulas $\Leftrightarrow$ pp-formulas on $\mathbb{A}$.

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6.2.8

6.2.2

6.1.43

(1) $I(\mathbb{B})$ contains all finite structures;

(5) $I(\mathbb{B})$ contains for $\forall A$ finite some $\mathbb{A}$ with $\mathrm{Pol}(\mathbb{A}) = \{\text{projections on } A\} =: \mathrm{Proj}_A$

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3.2.2

(8) $I(\mathbb{B})$ contains $\mathbb{A}$ with $|A| \geq 2$ s.t. $\mathrm{Pol}(\mathbb{A}) = \mathrm{Proj}_A$.

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Proof: We prove $(1) \Rightarrow (5) \Rightarrow (6)$ and $(7) \Rightarrow (8)$.

(1) $I(\mathbb{B})$ contains all finite structures;

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Proof: We prove (1) $\Rightarrow$ (5) $\Rightarrow$ (6) and (7) $\Rightarrow$ (8).

(1) $\Rightarrow$ (5): $\mathbb{A} := (A; \{(x,y,z) \mid x = y \vee y = z\}, \{a\} : a \in A)$

(5) $\Rightarrow$ (6): Theorem we saw $\Rightarrow \underline{A} \in Exp\, HSP^{fin}(\underline{B})$ $\{\text{operations of } \underline{A}\} = Pol(\mathbb{A})$

(7) $\Rightarrow$ (8): Reverse the same theorem.