Section 6.4 - Reflections

Mittwoch, 21. Februar 2024 15:00

- black text: this stems from the book
- blue text: deviates from book bc of typos etc

Def: $\underline{B}$ τ-algebra, A set, $h: B \to A$ and $g: A \to B$. The reflection of $\underline{B}$ w.r.t. g and h is τ-algebra $\underline{A}$ where for all $f \in \tau$ of arity n and all $x_1, \dots, x_n \in A$,

$$f^{\underline{A}}(x_1, \dots, x_n) := h(f^{\underline{B}}(g(x_1), \dots, g(x_n))$$

For a class $\mathcal{C}$ of τ-algebras, the class of their reflections is denoted by $\mathrm{Refl}(\mathcal{C})$.

wonderland: for every operation f of $\underline{B}$ then: apply Refl to operation clones. Need this at one point in proof of the theorem.

Analogous to the HSP-Lemma, the following holds:

Lemma $\mathcal{C}$ class of τ-algebras.

- The smallest class of τ-algebras containing $\mathcal{C}$ and closed under Refl, H, S and P is $\mathrm{Refl}\,P(\mathcal{C})$.
- The smallest class of τ-algebras containing $\mathcal{C}$ and closed under Refl, H, S and P^{fin} is $\mathrm{Refl}\,P^{fin}(\mathcal{C})$.

Proof: We show first statement. Suffices to show closedness of $\mathrm{Refl}\,P(\mathcal{C})$ under Refl, H, S, P.

Refl: By the definition of Reflection, the following diagram commutes.

$$\begin{tikzcd}
\underline{B} \arrow[r, shift left, "h_1"] \arrow[r, phantom, "\mathbf{Refl}"] \arrow[rr, bend right=40, "h_2\circ h_1"'] & \underline{A} \arrow[l, shift left, "g_1"] \arrow[r, shift left, "h_2"] \arrow[r, phantom, "\mathbf{Refl}"] & \underline{C} \arrow[l, shift left, "g_2"] \arrow[ll, bend left=60, "g_1\circ g_2"]
\end{tikzcd}$$

$\sim h_2(h_1(f^{\underline{A}}(g_1(g_2(x)))))$

$\mathbf{Refl}$

$\sim (h_2 \circ h_1)(f^{\underline{A}}((g_1 \circ g_2)(x)))$

Thus, $\mathrm{Refl}(\mathrm{Refl}(\mathcal{K})) = \mathrm{Refl}(\mathcal{K})$ for any class $\mathcal{K}$.

H: Show that $H(\mathcal{K}) \subseteq \mathrm{Refl}(\mathcal{K})$ for any class $\mathcal{K}$.

Given $\underline{A} \in H(\mathcal{K})$, there is a surjective homomorphism h from some $\underline{B} \in \mathcal{K}$ to $\underline{A}$. Let $g: A \to B$ s.t. $h \circ g = id_A$. It holds for $f \in \tau$ of arity n and $x_1, \dots, x_n \in A$ that

$$h(f^{\underline{B}}(g(x_1), \dots, g(x_n)) = f^{\underline{A}}(h\circ g(x_1), \dots, h \circ g(x_n)) = f^{\underline{A}}(x_1, \dots, x_n).$$

Thus, $\underline{A}$ is the reflection of $\underline{B}$ w.r.t. h and g, so $\underline{A} \in \mathrm{Refl}(\mathcal{K})$.

S: Show that $H(\mathcal{K}) \subseteq \mathrm{Refl}(\mathcal{K})$ for any class $\mathcal{K}$.

Given $A \leq B \in \mathcal{C}$, set $g: A \to B$ to be the identity and $h: B \to A$ any extension of g to B. Then $\underline{A}$ is the reflection of $\underline{B}$ w.r.t. h and g, analogous to H.

P: Show that $P(\mathrm{Refl}\,P(\mathcal{K})) \subseteq \mathrm{Refl}((P(\mathcal{K}))$.

Given a set I, algebras $(\underline{B}_i)_{i\in I} \in (P(\mathcal{K}))^I$ and τ-algebras $(A_i)_{i \in I}$ s.t. for all i, A_i is the reflection of B_i w.r.t. h_i and g_i.

Then $h: (b_i)_{i\in I} \mapsto (h_i(b_i))_{i \in I}$ and $g: (a_i)_{i\in I} \mapsto (g_i(a_i))$ witness for all i, A_i is the reflection of B_i w.r.t. h_i and g_i.

Then $h: (b_i)_{i\in I} \mapsto (h_i(b_i))_{i\in I}$ and $g: (a_i)_{i\in I} \mapsto (g_i(a_i))$ witness that $\prod_{i\in I} A_i$ is a reflection of $\prod_{i\in I} B_i$, which proves

$P(\mathrm{Refl}\, P(\mathcal{K})) \subseteq \mathrm{Refl}\, P(\mathcal{K})$. $\square$

Notation for a class $\mathcal{C}$ of relational structures, write

$H(\mathcal{C})$ for the class of all homomorphically equivalent structures.

$\mathrm{Red}(\mathcal{C})$ for the class of all pp-reducts $\mathbb{A}$ of structures $\mathbb{B}$ in $\mathcal{C}$, i.e. same domain and all relations pp-definable.

$I(\mathcal{C})$ for the class of structures with a pp-interpretation in a structure from $\mathcal{C}$.

Remark Chapter 3.6: Given a class $\mathcal{C}$ of structures, $HI(\mathcal{C})$ is the class of structures that can be pp-constructed from $\mathcal{C}$.

Theorem: Let $\mathbb{B}, \mathbb{C}$ be at most countable ω-categorical structures and let $\underline{C}$ be a polymorphism algebra of $\mathbb{C}$. Then:

i) $\mathbb{B} \in HRed(\mathbb{C})$ iff there exists $\underline{B} \in \mathrm{Exp}\,\mathrm{Refl}(\underline{C})$ s.t. $\mathrm{Clo}(\underline{B}) = \mathrm{Pol}(\mathbb{B})$

ii) $\mathbb{B} \in HI(\mathbb{C})$ iff there exists $\underline{B} \in \mathrm{Exp}\,\mathrm{Refl}\,P^{fin}(\underline{C})$ s.t. $\mathrm{Clo}(\underline{B}) = \mathrm{Pol}(\mathbb{B})$

Proof: i) "$\Rightarrow$" Let $\mathbb{C}' \in \mathrm{Red}(\mathbb{C})$ s.t. $\mathbb{B} \in H(\mathbb{C}')$ witnessed by $h: \mathbb{C}' \to \mathbb{B}$ as well as $g: \mathbb{B} \to \mathbb{C}'$. Set $\underline{C}'$ to be the expansion of $\underline{C}$ obtained by adding $\mathrm{Pol}(\mathbb{C}') \setminus \mathrm{Clo}(\underline{C})$ to the signature, making $\underline{C}'$ a polymorphism algebra of $\mathbb{C}'$.

$$\begin{tikzcd} B \arrow[r, bend left, "h"] & A \arrow[l, bend left, "g"] \end{tikzcd}$$

Let $\underline{B}'$ be the reflection of $\underline{C}'$ w.r.t. h and g.

All operations on $\underline{B}'$ are obtained as compositions of operations of the form $f^{\underline{B}'}(x_1,\dots,x_n) := h(f^{\underline{C}'}(g(x_1),\dots,g(x_n)))$. As h, g are homomorphisms and $f^{\underline{C}'} \in \mathrm{Pol}(\mathbb{C}')$, they preserve all relations of $\mathbb{B}$, which gives $\mathrm{Clo}(\underline{B}') \subseteq \mathrm{Pol}(\mathbb{B})$.

which gives $Clo(\underline{B}') \subseteq Pol(\mathbb{B})$.

Extending $\underline{B}'$ by adding $Pol(\mathbb{B}) \setminus Clo(\underline{B}')$ to it's signature gives us $\underline{B} \in Exp(\underline{B}') \subseteq Exp\,Refl(\underline{C}')$

$$\subseteq Exp\,Refl\,Exp(\underline{C})$$
$$= Exp\,Refl(\underline{C})$$

"$\mathbb{B} \in HRed(\mathbb{C}) \Leftarrow$ there exists $\underline{B} \in Exp\,Refl(\underline{C})$ s.t. $Clo(\underline{B}) = Pol(\mathbb{B})$"

Suppose that the reflection $\underline{B}$ of $\underline{C}$ w.r.t. $h: C \to B$ and $g: B \to C$ is such that $Clo(\underline{B}) \subseteq Pol(\mathbb{B})$.

Set $\mathbb{C}'$ to be the structure with domain C, the same signature as $\mathbb{B}$ and for all k-ary rel.symb. R of $\mathbb{B}$ the relation

$$R^{\mathbb{C}'} := \{ (f(g(b_1^1),\dots,g(b_1^\ell)),\dots,f(g(b_k^1),\dots,g(b_k^\ell))) \mid \ell \in \mathbb{N}, f \in Pol(\mathbb{C})^{(\ell)}, \overline{b^1},\dots,\overline{b^\ell} \in R^{\mathbb{B}} \}$$

These relations are preserved by $Pol(\mathbb{C})$ and therefore pp-def. in $\mathbb{C} \Rightarrow \mathbb{C}' \in Red(\mathbb{C})$.

As $Pol(\mathbb{C})$ contains the projections, g is a hom from $\mathbb{B}$ to $\mathbb{C}'$.

h is a homomorphism from $\mathbb{C}'$ to $\mathbb{B}$:

Let $f(g(b^1),\dots,g(b^\ell)) \in R^{\mathbb{C}'}$ (non-unary case works analogously)

then $\varphi: (x^1,\dots,x^\ell) \mapsto h(f(g(x^1),\dots,g(x^\ell)))$ is an operation on $\underline{B} \in Refl(\underline{C})$. Since $Clo(\underline{B}) \subseteq Pol(\mathbb{B})$, φ is a polymorphism and thus $h(f(g(b^1),\dots,g(b^\ell))) \in R^{\mathbb{B}}$.

Thus, we have $\mathbb{B} \in H(\mathbb{C}') \subseteq H(Red(\mathbb{C}))$.

ii) "$\mathbb{B} \in HI(\mathbb{C}) \Rightarrow$ there exists $\underline{B} \in Exp\,Refl\,P^{fin}(\underline{C})$ s.t. $Clo(\underline{B}) = Pol(\mathbb{B})$"

Let $\mathbb{B} \in HI(\mathbb{C})$.

$\Rightarrow \exists\ \mathbb{D} \in I(\mathbb{C})$ s.t. $\mathbb{B} \in H(\mathbb{D})$

6.3.7 $\Rightarrow \exists\ \underline{D} \in Exp\,HSP^{fin}(\underline{C})$ s.t. $Clo(\underline{D}) = Pol(\mathbb{D})$

$\overset{i)}{\Rightarrow} \exists\ \underline{B} \in \mathrm{ExpRefl}(\underline{D})$ s.t. $Clo(\mathbb{B}) = Pol(\underline{B})$

$\underline{B} \in \mathrm{ExpRefl}(\underline{D}) \subseteq \mathrm{Exp}\ \mathrm{Refl}(\mathrm{Exp}\,HSP^{fin}(\underline{C}))$
$= \mathrm{Exp}\ \mathrm{Refl}\ P^{fin}(\underline{C})$.

ii) "$\mathbb{B} \in HI(\mathbb{C}) \Leftarrow$ there exists $\underline{B} \in \mathrm{Exp}\ \mathrm{Refl}\ P^{fin}(\underline{C})$ s.t. $Clo(\underline{B}) = Pol(\mathbb{B})$"

This part can currently only be found in "Graph Homomorphisms and Universal Algebra".

Suppose that there exists $\underline{B} \in \mathrm{ExpRefl}\,P^{fin}(\underline{C})$ s.t. $Clo(\underline{B}) = Pol(\mathbb{B})$.

$\Rightarrow \exists\ \underline{D} \in P^{fin}(\underline{C})$ s.t. $\underline{B} \in \mathrm{ExpRefl}(\underline{D})$. Let $n \in \mathbb{N}$ s.t. $\underline{D} = \underline{C}^n$

Set $\mathbb{D}$ to be the structure with domain C^n with relations $Inv(Clo(\underline{D}))$. It follows that $\mathbb{D}$ has an n-dimensional pp-interpretation in $\mathbb{C}$ and by Prop. 6.1.5 and *, $Pol(\mathbb{D}) = Clo(\underline{D})$.

By i), $\mathbb{B} \in H\,Red(\mathbb{D}) = H\,Red\,P^{fin}(\mathbb{C}) \subseteq HI(\mathbb{C})$. □

*Note that as $Clo(\underline{C})$ is locally closed (by Cor. 6.1.6), so is $Clo(\mathbb{D})$:

$f \in \overline{Clo(\mathbb{D})}^{(k)} \Rightarrow \forall\ F \subseteq \mathbb{D}^k$ finite $\exists g \in Clo(\mathbb{D})^{(k)}$: $f|_F = g|_F$.

g acts componentwise like some $\tilde{g} \in Clo(\mathbb{C})^{(k)}$

$\Rightarrow f$ acts componentwise and the same for all components.

$\Rightarrow \forall i \in \{1,\dots,n\}$: $\pi_i \circ f \circ \left[(x_1,\dots,x_k) \mapsto (\underbrace{(x_1,\dots,x_k),\dots,(x_1,\dots,x_k)}_{n})\right]$
is equal to some $\tilde{h} \in Clo(\mathbb{C})^{(n)}$ as $Clo(\mathbb{C})$ is closed.

$\Rightarrow f$ acts like $\tilde{h}$ on all components.

Corollary: Let $\mathbb{B}$ be an at most countable, ω-categorical structure and let $\underline{B}$ be a polymorphism algebra of $\mathbb{B}$. TFAE:

i) $HI(\mathbb{B})$ contains $\mathbb{K}_3$;

ii) $HI(\mathbb{B})$ contains all finite structures;

iii) $HI(\mathbb{B})$ contains $(\{0,1\}; 1IN3)$;

iii) $HI(\mathbb{B})$ contains $(\{0,1\}; 1\text{IN}3)$;

iv) $\mathrm{Refl}\, P^{fin}(\underline{B})$ contains an algebra of size ≥ 2 all of whose operations are projections;

v) $\mathrm{Refl}\, P^{fin}(\underline{B})$ contains for every finite set A an algebra on A all of whose operations are projections.

If these conditions apply, $\mathbb{B}$ has a finite-signature reduct with an NP hard CSP.

Proof: i) ⇒ ii) $I(\mathbb{K}_3)$ contains all finite structures (Cor. 3.2.1) and $IHI(\mathbb{B}) \subseteq HI(\mathbb{B})$ (Th 3.6.2)

ii) ⇒ iii) trivial

iii) ⇒ iv) The polymorphisms of $1\text{IN}3$ only contain projections.

Theorem $\Rightarrow \exists\, \underline{A} \in \cancel{\mathrm{Exp}}\,\mathrm{Refl}\, P^{fin}(\underline{B}) : \mathrm{Clo}(\underline{A}) \overset{=}{\subseteq} \mathrm{Pol}(1\text{IN}3)$

iv) ⇒ v) Let $\underline{A} \in \mathrm{Refl}\, P^{fin}(\underline{B})$ s.t. $|A| \geq 2$ and $\mathrm{Clo}(\underline{A}) \subseteq \mathrm{Proj}_A$

Th.6.3.10 $\Rightarrow HSP^{fin}(\underline{A})$ contains for every finite set S an alg. $\underline{S}$ on S s.t. $\mathrm{Clo}(\underline{S}) \subseteq \mathrm{Proj}_S$.

Now $\underline{S} \in HSP^{fin}(\underline{A}) \subseteq HSP^{fin}(\mathrm{Refl}\, P^{fin}(\underline{B})) \overset{\text{Lemma}}{\subseteq} \mathrm{Refl}\, P^{fin}(\underline{B})$

v) ⇒ i) Let $\underline{A} \in \mathrm{Refl}\, P^{fin}(\underline{B})$ s.t. $A = \{0,1,2\}$ and $\mathrm{Clo}(\underline{A}) \subseteq \mathrm{Proj}_A$. Then $\mathrm{Clo}(\underline{A}) \subseteq \mathrm{Pol}(\mathbb{K}_3)$

$\Rightarrow \exists\, \underline{C} \in \mathrm{Exp}\,\mathrm{Refl}\, P^{fin}(\underline{B})$ s.t. $\mathrm{Clo}(\underline{C}) = \mathrm{Pol}(\mathbb{K}_3)$

$\overset{\text{Theorem}}{\Rightarrow} \mathbb{K}_3 \in HI(\mathbb{B})$

The hardness follows from Cor. 3.7.1 ($\mathbb{K}_3 \in HI(\mathbb{B}) \Rightarrow \exists$ finite sign. reduct whose CSP is NP-hard). □