Birkhoff's HSP Thm and related stuff

Def. If τ functional signature, an identity (over τ) is τ-sent. of the form:

$$\forall x_1 \cdots \forall x_n \; s(x_1, \dots, x_n) = t(x_1, \dots, x_n). \qquad (s, t \;\; \tau\text{-terms})$$

So $t^{\mathbb{A}} = s^{\mathbb{A}}$ iff $\mathbb{A} \models \forall \bar{x} \; s(\bar{x}) = t(\bar{x})$.

Birkhoff's Theorem (6.5.1). Let $\mathbb{A}, \mathbb{B}$ τ-algebras. TFAE:

(1) Every identity that holds in $\mathbb{B}$ holds in $\mathbb{A}$.

(2) $\mathbb{A} \in HSP(\mathbb{B})$.

If $\mathbb{A}, \mathbb{B}$ are finite can add:

(3) $\mathbb{A} \in HSP^{fin}(\mathbb{B})$.

Birkhoff's Theorem. TFAE:

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Take: $I = \{ (\bar{a}, \bar{b}) : \bar{a} \in A^n$ inj., $\bar{b} \in B^n$, $n \in \mathbb{N} \}$

If $\mathbb{A}, \mathbb{B}$ finite I can be chosen to be fin.

proof. (2) $\Rightarrow$ (1): ✓

(1) $\Rightarrow$ (2) Choose big enough index set I s.t. there is injective map $c: A \to B^I$ with:

$$\forall \bar{a} \in A^n \text{ injective}, \forall \bar{b} \in B^n \ \exists i \in I : (c(a_1)_i, \dots, c(a_n)_i) = \bar{b}. \qquad (*)$$

Let $\mathbb{C} \leq \mathbb{B}^I$ subalgebra gen. by $c(A)$. So $C = \{ t^{\mathbb{B}^I}(c(\bar{a})) : n \in \mathbb{N}, \bar{a} \in A^n \text{ inj.}, t \ \tau\text{-term} \}$

Define $\mu : C \to A$, $t(c(\bar{a})) \mapsto t(\bar{a})$ (t τ-term, $\bar{a} \in A^n$ inj.).

μ well-defined: If $s(c(\bar{a})) = t(c(\bar{a}))$, by $(*)$: $s^{\mathbb{B}} = t^{\mathbb{B}}$, so $s^{\mathbb{A}} = t^{\mathbb{A}}$ in particular $s(\bar{a}) = t(\bar{a})$.

Clearly μ surjective τ-homom., so $\mathbb{A} \in HSP(\mathbb{B})$. $\square$

Corollary. A class of τ-structures is a variety iff it is axiomatized by a set of identities over τ.

pf. $\mathcal{V}$ variety ; find $\mathbb{B} \in \mathcal{V}$: $HSP(\mathbb{B}) = \mathcal{V}$.

Theorem. (Cor. 9.2.8 [1]) Let $\mathcal{L}$ any signature. A class of $\mathcal{L}$-structures is closed under products, substructures and homomorphic images iff it is axiom. by a set of sent. of the form: $\forall \bar{x}\, \varphi(\bar{x})$, φ atomic.

Theorem. (Cor 9.5.10 [1]) Let $\mathcal{L}$ any signature. A class $\mathcal{K}$ of $\mathcal{L}$-struct. is axiomatized by a set of $\mathcal{L}$-sentences iff it is closed under ultraproducts, isomorphic copies and ultraroots ($A \in \mathcal{K}$ if some ultrapower of A lies in $\mathcal{K}$).

$A \equiv B$ iff $A^I/\mathcal{U} \cong B^I/\mathcal{U}$

Keisler-Shelah

[1] Model Theory - Hodges

Let $\mathbb{A}$, $\mathbb{B}$ τ-algebras. The map, the natural homom. $\mathrm{Clo}(\mathbb{B}) \to \mathrm{Clo}(\mathbb{A})$

$$\mathrm{Clo}(\mathbb{B}) \to \mathrm{Clo}(\mathbb{A}), \quad t^{\mathbb{B}} \mapsto t^{\mathbb{A}} \quad (t \text{ some } \tau\text{-term})$$

is well defined iff for all τ-terms s, t: $s^{\mathbb{B}} = t^{\mathbb{B}} \Rightarrow s^{\mathbb{A}} = t^{\mathbb{A}}$.

<u>Thm. (6.5.10.)</u> $\mathbb{A}, \mathbb{B}$ τ-algebras. TFAE:

(1) The natural homom. $\mathrm{Clo}(\mathbb{B}) \to \mathrm{Clo}(\mathbb{A})$ exists.

(2) All identities that hold in $\mathbb{B}$ also hold in $\mathbb{A}$.

(3) $\mathbb{A} \in HSP(\mathbb{B})$.

If $\mathbb{A}, \mathbb{B}$ are finite we can add:

(4) $\mathbb{A} \in HSP^{fin}(\mathbb{B})$. $\square$

<u>Cor. (6.5.11)</u>. $\mathbb{C}$, $\mathbb{D}$ op. clones. There is surjective hom. $\mathbb{D} \to \mathbb{C}$ iff there are algebras $\mathbb{A}, \mathbb{B}$ in same signature with $\mathbb{A} \in HSP(\mathbb{B})$ and $\mathrm{Clo}(\mathbb{A}) = \mathbb{C}$, $\mathrm{Clo}(\mathbb{B}) = \mathbb{D}$.

<u>Pf</u>. $\Rightarrow$: Take signature $\mathbb{D}$ to build $\mathbb{A}, \mathbb{B}$. Apply 6.5.10; $\Leftarrow$: Apply 6.5.10.

A(n) (abstract) clone is a multisorted structure

$$\mathbb{C} = \left(C^{(1)}, C^{(2)}, \dots ; (pr_i^k)_{1\le i\le k}, (comp_\ell^k)_{k,\ell\ge 1} \right)$$

"2-ary operations of $\mathbb{C}$" — const. symb. — function symbols

where:
- $pr_i^k \in C^{(k)}$
- $comp_\ell^k : C^{(k)} \times \overbrace{C^{(\ell)} \times \dots \times C^{(\ell)}}^{k\text{-times}} \to C^{(\ell)}, \quad (f, g_1, \dots, g_k) \mapsto f \circ (g_1, \dots, g_k)$

s.t.
- $f \circ (p_1^k, \dots, p_k^k) = f$
- $pr_i^k \circ (f_1, \dots, f_k) = f_i$
- $(f \circ (g_1, \dots, g_k)) \circ (h_1, \dots, h_\ell) = f \circ (g_1 \circ (h_1, \dots, h_\ell), \dots, g_k \circ (h_1, \dots, h_\ell))$

Every operation clone can be interpreted to be a clone in a straightforward way.

A **homomorphism** $\varphi : \mathbb{C} \to \mathbb{D}$ of clones is a collection of maps $(\varphi_1, \varphi_2, \dots)$ such that:

- $\varphi_k : C^{(k)} \longrightarrow D^{(k)}$
- $\varphi_k(p_i^k) = p_i^k$
- $\varphi_\ell(f \circ (g_1, \dots, g_k)) = \varphi_k(f) \circ (\varphi_\ell(g_1), \dots, \varphi_\ell(g_k))$

An **isomorphism** $\varphi : \mathbb{C} \to \mathbb{D}$ is a homom. s.t. there is a homom. $\psi : \mathbb{D} \to \mathbb{C}$ with $\psi \circ \varphi = id_{\mathbb{C}}$, $\varphi \circ \psi = id_{\mathbb{D}}$.

<u>Remark</u>. (Cayley's Thm.) Every clone is isomorphic to an operation clone.

Pf. Given clone $\mathbb{C}$, let $X = \prod_i C^{(i)}$. For $f \in C^{(k)}$ define:

$$\varphi(f) : X^k \to X, \quad (c_1, \dots, c_k) \mapsto \big(\overbrace{f \circ (c_1(i), \dots, c_k(i))}^{\in C^{(i)}}\big)_i .$$

Easy to check that $\varphi : \mathbb{C} \to \mathcal{O}_X$, $f \mapsto \varphi(f)$ isomorph. onto its image. $\square$

Given τ-algebra $\mathbb{A}$, and identity $\forall \bar{x}\ s(\bar{x}) = t(\bar{x})$, want to express:

$$\mathbb{A} \models \forall \bar{x}\ s(\bar{x}) = t(\bar{x})$$

using the multisorted structure $\mathrm{Clo}(\mathbb{A})$.

<u>Example</u>. Let $f, g \in \tau$: $\mathbb{A} \models \forall x_1 \forall x_2\ \overbrace{f(x_1, x_2) = g(x_2, x_1)}^{\psi(x_1, x_2)}$ iff

$$\underbrace{\mathrm{Clo}(\mathbb{A}) \models f^{\mathbb{A}} \circ (pr^2_1, pr^2_2) = g^{\mathbb{A}} \circ (pr^2_2, pr^2_1)}_{\psi^{\dagger}(f^{\mathbb{A}}, g^{\mathbb{A}})}.$$

<u>Generally</u>: If $\psi(x_1, \dots, x_n) := s(x_1, \dots, x_n) = t(x_1, \dots, x_n)$ eq. of τ-terms in which $f_1, \dots, f_k \in \tau$ appear there is clone formula $\psi^{\dagger}(z_1, \dots, z_k)$ s.t.

$$\mathbb{A} \models \forall \bar{x}\ \psi(\bar{x}) \iff \mathrm{Clo}(\mathbb{A}) \models \psi^{\dagger}(f_1^{\mathbb{A}}, \dots, f_k^{\mathbb{A}})$$

Def. An operation clone $\mathbb{C}$ staisfies a set of identities Σ over some signature τ, if there is τ-algebra $\mathbb{A} \models \Sigma$ with $\mathrm{Clo}(\mathbb{A}) \subseteq \mathbb{C}$.

Lemma (6.5.13). Let $\mathbb{C}$ operation clone over finite domain, Σ set of identities over τ. $\mathbb{C}$ satisfies Σ iff it satisfies every finite subset of Σ.

Given clone $\mathbb{C}$ denote by $\mathbb{C}_{\leq k}$ the subclone gen. by $C^{(1)} \cup \dots \cup C^{(k)}$.

Fact: If $\mathbb{C}, \mathbb{D}$ clones with $C^{(k)}$ finite for all k, s.t. $\mathbb{C}_{\leq k} \cong \mathbb{D}_{\leq k}$ then $\mathbb{C} \cong \mathbb{D}$.

Pf: For every k there are only finitely many isos : $\mathbb{C}_{\leq k} \to \mathbb{D}_{\leq k}$

Construct tree as follows: • Vertices on level k are isos: $\mathbb{C}_{\leq k} \to \mathbb{D}_{\leq k}$

• $\tilde{\sigma} : \mathbb{C}_{\leq k+1} \to \mathbb{D}_{\leq k+1}$ is child of $\sigma : \mathbb{C}_{\leq k} \to \mathbb{D}_{\leq k}$, if $\tilde{\sigma}|_{\mathbb{C}_{\leq k}} = \sigma$.

This is infinite tree with finite branching $\Rightarrow \exists$ infinite path, i.e. iso. $\mathbb{C} \to \mathbb{D}$. $\square$