Birkhoff's Thm. Part 2

Lemma (6.5.13). Let $\mathcal{C}$ operation clone over finite domain, Σ set of identities over τ. $\mathcal{C}$ satisfies Σ iff it satisfies every finite subset of Σ.

(τ-sent. of the form: $\forall \bar{x}\ f(\bar{x}) = g(\bar{x})$, f, g terms

Recall: $\mathcal{C}$ satisfies set of identities Σ, if there is τ-algebra $\mathbb{A}$ with $\mathbb{A} \models \Sigma$ and $\mathrm{Clo}(\mathbb{A}) \cong \mathcal{C}$.

Last time: $\mathcal{C}, \mathcal{D}$ clones with $\mathcal{C}^{(k)}$ finite and $\underbrace{\mathcal{C}_{\leq k} \cong \mathcal{D}_{\leq k}\ \forall k}_{\text{"}\mathcal{C} \text{ and } \mathcal{D} \text{ are locally isomorphic"}}$
$\Rightarrow \mathcal{C} \cong \mathcal{D}$.

Recall: $\mathcal{C}_{\leq k}$ denotes subclone generated by $C^{(1)} \cup \dots \cup C^{(k)}$.

If $\mathcal{C}$ and $\mathcal{D}$ are arbitrary clones, being locally isomorphic does not imply being isomorphic:

Essential k-ary op. of $\mathcal{C}$: $\{ c_n^{(k)} : n \in \mathbb{N}_{\geq k} \}$. (And identity, if $k=1$)

Essential k-ary op. of $\mathcal{D}$: $\{ c_n^{(k)} : n \in \mathbb{N}_{\geq k} \cup \{\infty\} \}$. (An id., if $k=1$)

Moreover there is a single const. k-ary operation $0^{(k)}$ for all $k \in \mathbb{N}$

Essential operations of $\mathcal{C}, \mathcal{D}$:

$$\mathcal{C}^{(3)}: \qquad c_3^{(3)}, \ldots$$

$$\mathcal{C}^{(2)}: \qquad c_2^{(2)}, \quad c_3^{(2)}, \ldots$$

$$\mathcal{C}^{(1)}: \mathrm{id}, \quad c_1^{(1)}, \quad c_2^{(1)}, \quad c_3^{(1)}, \ldots$$

$$\mathcal{D}^{(3)}: \qquad c_3^{(3)}, \ldots, c_\infty^{(3)}$$

$$\mathcal{D}^{(2)}: \qquad c_2^{(2)}, \quad c_3^{(2)}, \ldots, c_\infty^{(2)}$$

$$\mathcal{D}^{(1)}: \mathrm{id}, \quad c_1^{(1)}, \quad c_2^{(1)}, \quad c_3^{(1)}, \ldots, c_\infty^{(1)}$$

Essential k-ary op. of $\mathcal{C}$: $\{ c_n^{(k)} : n \in \mathbb{N}_{\geq k} \}$. (And identity, if $k=1$)
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Moreover there is a single const. k-ary operation $0^{(k)}$ for all $k \in \mathbb{N}$

<u>Composition</u>: • $c_n^{(k)}(x_1, \dots, x_k) = c_n^{(k)}(x_{\sigma(1)}, \dots, x_{\sigma(k)})$ $\forall n \geq k$ $\forall \sigma \in S_k$.

• $(x_1, \dots, x_{k-1}) \mapsto c_n^{(k)}(x_{i_1}, \dots, x_{i_k}) = c_n^{(k-1)}$ $\forall n \geq k$, if $\{i_1, \dots, i_k\} = \{1, \dots, k-1\}$

• $c_n^{(k)}(g_1, \dots, g_k) = 0^{(\ell)}$ $\forall n \geq k$, if $g_i = c_m^{(\ell)}$ for some i, ℓ, m.

In $\mathcal{C}_{\leq k}, \mathcal{D}_{\leq k}$ all $c_n^{(k)}$ behave in the same way. So any biject. $\alpha: \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ with $\alpha|_{\{1, \dots, k-1\}} = \mathrm{id}$ induces isomorphism:

$$\mathcal{C}_{\leq k} \to \mathcal{D}_{\leq k}, \quad c_i^{(\ell)} \mapsto c_{\alpha(i)}^{(\ell)}.$$

$\Rightarrow \mathcal{C}, \mathcal{D}$ locally isomorphic.

$\mathcal{C}^{(3)}$: $\qquad c_3^{(3)}, \dots$ $\qquad \mathcal{D}^{(3)}$: $\qquad c_3^{(3)}, \dots, c_\infty^{(3)}$

$\mathcal{C}^{(2)}$: $\qquad c_2^{(2)}, c_3^{(2)}, \dots$ $\qquad \mathcal{D}^{(2)}$: $\qquad c_2^{(2)}, c_3^{(2)}, \dots, c_\infty^{(2)}$

$\mathcal{C}^{(1)}$: $\mathrm{id}, c_1^{(1)}, c_2^{(1)}, c_3^{(1)}, \dots$ $\qquad \mathcal{D}^{(1)}$: $\mathrm{id}, c_1^{(1)}, c_2^{(1)}, c_3^{(1)}, \dots, c_\infty^{(1)}$

Why $\mathcal{C} \not\cong \mathcal{D}$?

In $\mathcal{D}$ there is seq. of essential operations $(c_\infty^{(k)})_{k\in\mathbb{N}}$ with:

$$c_\infty^{(k+1)}(x_1, \dots, x_k, x_k) = c_\infty^{(k)}, \quad \forall k.$$

In $\mathcal{C}$ there is no such chain. $\Rightarrow \mathcal{C} \not\cong \mathcal{D}$.

<u>Lemma (6.5.13)</u>. $\mathscr{C}$ operation clone over finite domain, Σ set of identities over τ. $\mathscr{C}$ satisfies Σ iff it satisfies every finite subset of Σ.

pf. $\Rightarrow$ is trivial. $\Leftarrow$: Let $\mathcal{L}$ be language of $\mathscr{C}$ (viewed as abstr. clone) together with const. symbol c_f for every $f \in \tau$ that appears somewhere in Σ. View $\mathscr{C}$ as an $\mathcal{L}$-structure.

Let $T = Th_{\mathcal{L}}(\mathscr{C})$. Let $S = \{\psi^{+}(c_{f_1}, \dots, c_{f_k}) : \forall \bar{x}\, \psi(\bar{x}) \in \Sigma$ built from $f_1, \dots, f_k \in \tau\}$

Recall:

$$A \models \forall \bar{x}\, \psi(\bar{x}) \Leftrightarrow Clo(A) \models \psi^{+}(f_1^A, \dots, f_k^A)$$

Have: $\mathscr{C} \models T \cup F$ for all finite $F \subseteq S$. Compactness: $\exists\, \mathbb{M} \models T \cup S$.

Let $\mathcal{D} := \mathbb{M}|_{\bigcup_i M_i}$. Easy to check: $\mathcal{D}$ is an abstract clone s.t. $\mathcal{D} \models S$. I.e. $\mathcal{D}$ satisfies Σ.

One can also check that: $\mathcal{D}_{\leq k} \cong \mathscr{C}_{\leq k}$ $\forall k$ (Reason: If $\mathcal{M} \equiv \mathcal{N}$ finite then $\mathcal{M} \cong \mathcal{N}$)

$\mathscr{C}$ clone over finite domain $\Rightarrow$ $\mathscr{C} \cong \mathcal{D}$. $\square$

<u>Corollary (6.5.14)</u>. $\mathcal{C}, \mathcal{D}$ clones. If $\mathcal{D}$ is the clone from a finite algebra, then there is clone homom. $\mathcal{C} \to \mathcal{D}$ iff for all pp-sentences σ: $\mathcal{C} \models \sigma \Rightarrow \mathcal{D} \models \sigma$.

<u>proof</u>. "$\Rightarrow$": ✓. "$\Leftarrow$": By Cayley's Thm. $\mathcal{C} = \mathrm{Clo}(\mathbb{A})$ for some τ-algebra $\mathbb{A}$.

Let Σ set of identities that hold in $\mathbb{A}$. Given finite $\Delta \subseteq \Sigma$ with identities built from $t_1, \dots, t_k \in \mathcal{T}$ there is clone formula $\psi_\Delta^\dagger(x_1, \dots, x_k)$ s.t.

$$\forall \tau\text{-algebras } \mathbb{B}: \quad \mathbb{B} \models \Delta \iff \mathrm{Clo}(\mathbb{B}) \models \psi_\Delta^\dagger(t_1^{\mathbb{B}}, \dots, t_k^{\mathbb{B}}).$$

$\Rightarrow$ $\mathrm{Clo}(\mathbb{A}) \models \psi_\Delta^\dagger(t_1^{\mathbb{A}}, \dots, t_k^{\mathbb{A}})$, so by assumption: $\mathcal{D} \models \exists x_1, \dots, x_k \, \psi_\Delta^\dagger(x_1, \dots, x_n)$.

$\Rightarrow$ $\mathcal{D}$ satisfies Δ; previous Lemma implies, that $\mathcal{D}$ satisfies Σ.

I.e. there is τ-algebra $\mathbb{B}$ with $\mathcal{D} \cong \mathrm{Clo}(\mathbb{B})$ and $\mathbb{B} \models \Sigma$.

$\Rightarrow$ The natural homom. $\underset{\mathcal{C}}{\underset{\cong}{\mathrm{Clo}(\mathbb{A})}} \to \underset{\mathcal{D}}{\underset{\cong}{\mathrm{Clo}(\mathbb{B})}}$, $t^{\mathbb{A}} \mapsto t^{\mathbb{B}}$ exists.

$\square$