Locally isomorphic $\neq$ globally isomorphic

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Defitniton. Two clones $\mathscr{C}, \mathscr{D}$ are said to be *locally isomorphic*, if for every $k \in \mathbb{N}$ the respective subclones generated by the k-ary operations are isomorphic, i.e. if (by slight abuse of notation)

$$\forall k \in \mathbb{N} \colon \mathscr{C}^{(k)} \cong \mathscr{D}^{(k)}.$$

If one restricts attention to clones whose k-ary operations are finite the notions of locally isomorphic and globally isomorphic (the 'globally' just serves to contrast locally: two clones are said to be globally isomorphic iff they are isomorphic as clones) coincide by a straightforward compactness argument. The following example, essentially due to Věra Trnková and Jiří Sichler [1] shows that in general, locally isomorphic clones need not be globally isomorphic.

The counterexample

Before describing the locally isomorphic clones, which are not globally isomorphic we fix some notation. For $k \in \mathbb{N}_0$, write $[k] = \{1, \ldots, k\}$, in particular $[0] = \emptyset$. Given some non-empty set X and a map $\alpha \colon [n] \to [k]$ let

$$\pi(\alpha) \colon X^k \to X^n,\, x \mapsto x \circ \alpha.$$

Here we view the set X^n as the set of maps $[n] \to X$. Note that $\pi(\alpha \circ \beta) = \pi(\beta) \circ \pi(\alpha)$.

Defitniton. Let $\mathscr{C}$ be a clone. A k-ary operation $f \in C^{(k)}$ is said to be *essential* if it cannot be written as $g \circ \pi(\alpha)$ for some $g \in \mathscr{C}^{(i)}$ and $\alpha \colon [i] \to [k]$ with $i < k$.

We now define the two clones $\mathscr{C}, \mathscr{D}$ as follows:

- $\mathscr{C}^{(0)} = \mathscr{D}^{(0)} = 0$.
- If $k \in \mathbb{N}$, the essential k-ary operations of $\mathscr{C}$ are the 'identity' and
$$\{c_i^{(k)} \colon (k, i) \in \mathbb{N}^2, k \leq i\}.$$
- If $k \in \mathbb{N}$, the essential k-ary operations of $\mathscr{D}$ are the unary identity $\pi(\mathrm{id}_{[1]})$ and
$$\{c_i^{(k)} \colon (k, i) \in \mathbb{N} \times \mathbb{N} \cup \{\infty\}, k \leq i\}.$$
- In addition to the essential operations both clones also contain the non-essential functions, in particular the projections.

Composition for $\mathscr{C}$ and $\mathscr{D}$ is defined as follows:

- Given surjective $\alpha \colon [n] \to [k]$, it holds $c_i^{(n)} \circ \pi(\alpha) = c_i^{(k)}$.
- For injective $\alpha, \beta \colon [k] \to [n]$, it holds $c_i^{(k)} \circ \pi(\alpha) = c_i^{(k)} \circ \pi(\beta)$ if and only if α and β have the same image.
- For $g_1, \ldots, g_k \in \mathscr{C}^{(n)}$, it holds $c_i^{[k]} \circ (g_1, \ldots, g_k) = 0 \circ \pi([0] \to [n])$ if one of the g_j is of the form $c_l^{[m]} \circ \pi(\alpha)$, for injective $\alpha \colon [m] \to [n]$.

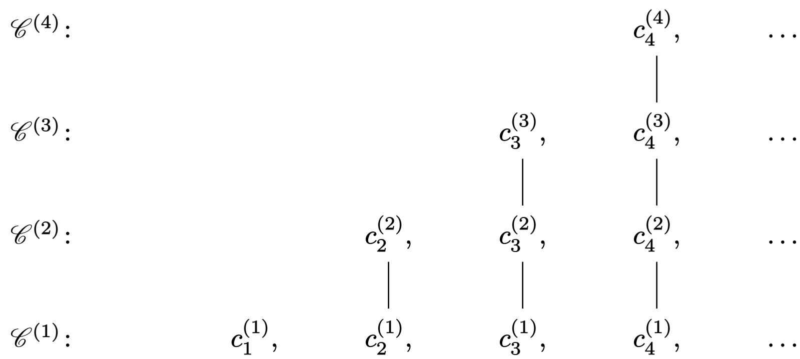

Figure 1: Essential operations of $\mathscr{C}$ and how they are related; the vertical line from e.g. $c_3^{(3)}$ to $c_3^{(2)}$ indicates that there is $\alpha\colon [3] \to [2]$ with $c_3^{(3)} \circ \pi(\alpha) = c_3^{(2)}$

Since any map $[n] \to [m]$ can be factored as a surjective map followed by an injective map, an easy inductive argument shows that every n-ary operation of $\mathscr{C}$ resp. $\mathscr{D}$ is of the form $f \circ \pi(\alpha)$, where f is an essential k-ary operation and $\alpha\colon [k] \to [n]$ an injective map.

Lemma. *The clones $\mathscr{C}$ and $\mathscr{D}$ are locally isomorphic.*

Proof. We will show that $\mathscr{C}^{(k)} \cong \mathscr{D}^{(k)}$ for arbitrary k. Pick an arbitrary bijection $\sigma\colon \mathbb{N} \to \mathbb{N} \cup \{\infty\}$, which restricts to the identity on $\{1, \ldots, k-1\}$. It is easy to check that the assignment

$$c_i^{(n)} \mapsto c_{\sigma(i)}^{(n)}$$

for $i \in \mathbb{N}$, $n \leq \min(i, k)$ induces an isomorphism of clones $\mathscr{C}^{(k)} \to \mathscr{D}^{(k)}$. □

Lemma. *The clones $\mathscr{C}$ and $\mathscr{D}$ are not isomorphic.*

Proof. For $k \in \mathbb{N}$ choose a surjective map $\alpha_k\colon [k+1] \to [k]$. The sequence $(c_\infty^{(k)})_k$ of essential operations in $\mathscr{D}$ has the property

$$\forall k \in \mathbb{N}\colon c_\infty^{(k+1)} \circ \pi(\alpha_k) = c_\infty^{(k)}.$$

In other words: There is an infinite chain of essential operations in $\mathscr{D}$, which 'stack' on top of each other. By design there is no such infinite chain in $\mathscr{C}$ and since isomorphisms of clones preserve 'infinite essential chains' there cannot be an isomorphism $\mathscr{C} \to \mathscr{D}$. □

References

[1] J. Sichler and V. Trnková. "On clones determined by their initial segments". en. In: *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 49.3 (2008), pp. 209–227. URL: http://www.numdam.org/item/CTGDC_2008__49_3_209_0/.