Reading Group

Fix τ.

Recall

$$\underline{A} \to \underline{B}$$

Def Let $\mathcal{C}$ be a class of τ-structures, which is closed under $\cong$. Then $CSP(\mathcal{C})$ is the problem decide for $\underline{A}$ if $\underline{A} \in \mathcal{C}$.

Fix $\underline{A}$.

$$\{\underline{B} \mid \underline{A} \to \underline{B}\}$$

$$= \{\underline{B} \mid \exists x_1 \dots x_n \, Q(\underline{A})\}$$

If $\underline{A}$ is finite, this problem is in P.

Fix $\underline{B}$.

Then $CSP(\underline{B})$ is the opposite Question.

If $\underline{B}$ is finite, $CSP(\underline{B})$ is in NP.

The satisfiability perspective

Def T Theory, φ sentence

- $T \models \varphi$ (T entails φ)
- $CSP(T)$: Find out for a p.p-sentence φ

if $T \cup \{\varphi\}$ is consistent.

Prop τ finite.

T a τ-theory,

$\underline{B}$ a τ-structure.

$$CSP(T) = CSP(\underline{B})$$

$\Leftrightarrow \forall \varphi$ p.p.: $T \models \neg\varphi \Leftrightarrow \underline{B} \not\models \varphi$.

Example

$$\underline{B} = (\mathbb{Z}, <)$$

T = linear orders

The existential second order perspective

CSP($\mathcal{C}$)

Def An ESO sentence Φ is a sentence of the form

$$\exists R_1, \dots R_m : \varphi$$

where $R_1, \dots, R_m$ are relations and φ is a f.o. sentence with signature $\tau \cup \{R_1, \dots, R_m\}$

CSP($\underline{B}$)

$\underline{B}$ finite

Here, we write $CSP(\Phi)$ for the problem find out for a model $\underline{A}$ if $\underline{A} \models \Phi$.

(This is not standard as $CSP(\Phi)$ is not needed to be closed under inverse homomorphisms or disjoint unions.)

Theorem (Fagin) $\mathcal{C}$ as above

$CSP(\mathcal{C}) = CSP(\Phi)$

for some ESO sentence Φ

$\Leftrightarrow$ $CSP(\mathcal{C})$ is in NP.

$$\begin{array}{c} CSP(\mathcal{C}) \\ | \\ ESO \,/\, NP \end{array}$$

Def An SNP sentence Φ is an ESO sentence

$$\exists R_1, \dots R_m :$$

$$\forall x_1, \dots, x_n \; \varphi$$

where φ is quantifier free.

Def An SNP sentence is called

- **monotone** if every literal of φ which has a symbol in $\tau \cup \{=\}$ is negative.
- **connected** if φ written as

$$\bigwedge_i \bigvee_j (\neg) R_{ij}(x_{ij1}, \dots, x_{ijn_{ij}})$$

defines a connected graph as follows: Whenever R_{ij} is negative: connect x_{ij1} to all x_{ijk} $\forall k \in \{1, \dots, n_{ij}\}$.

CSP(e)

$\supseteq$

ESO

|

SNP

MSNP CSNP

Lemma $\underline{A}$ infinite structure, Φ an SNP sentence

$$\underline{A} \models \Phi \iff \underline{A}' \models \Phi \quad \forall\, \underline{A}' \subseteq \underline{A} \text{ finite}$$

Proof $\Longrightarrow$ $\forall$ restricts to A'

$\Longleftarrow$ Compactness

Thm (Feder & Vardi)

Φ is SNP sentence. $\{\underline{A} \mid \underline{A} \models \Phi, \underline{A}$ finite$\}$ is closed under inverse homomorphisms

$\iff$ Φ is equivalent to a monotone SNP sentence.

Proof $\Leftarrow$

$\Rightarrow$ Φ Idea: add E, R_i'

$$\begin{array}{ccc} \underline{A} \models \Phi & \Longrightarrow & \underline{A}/_E \models \Phi \\ \Uparrow & & \Downarrow \\ \underline{A} \models \Phi & \Longleftarrow & \underline{A}/_E \models \Phi \end{array}$$

Thm

Φ is SNP sentence.

Then $\{\underline{A} \mid \underline{A} \models \Phi, \underline{A}$ finite$\}$ is closed under disjoint unions

$\Leftrightarrow$ Φ is equivalent to some connected SNP.

<u>Prop</u> Φ is an SNP sentence. TFAE:

(1) Φ is equivalent to both an MSNP and an CSNP sentence.

(2) Φ is equivalent to a sentence which is both MSNP and CSNP.

(3) There is an (infinite) structure $\underline{B}$ s.t. $CSP(\underline{\Phi}) = CSP(\underline{B})$.

CSP(e)

ESO / NP

SNP

CSNP MSNP

CMSNP
$\hat{=}$ SNP and CSP($\underline{B}$)
$\underline{B}$ infinite

MMSNP

CMMSNP

CSP($\underline{B}$),
$\underline{B}$ finite

second
M ... monadic (all $\exists R$ are
unary relations)

p-time
reduction