IDEMPOTENT ALGEBRAS & TAYLOR TERMS

Def: An operation $f: A^n \to A$, $n \geq 2$ is a Taylor operation if it satisfies a set $\{\varphi_1 \dots \varphi_n\}$ of Taylor identities i.e. φ_i is of the form:

$$\forall x,y: f(z_1 \dots z_n) = f(z_1' \dots z_n'),$$

where $z_1 \dots z_n, z_1' \dots z_n' \in \{x,y\}$, $z_i \neq z_i'$

i.e.:

$$f\begin{pmatrix} x & * & \cdots & * \\ * & x & * & \vdots \\ \vdots & * & \ddots & * \\ * & \cdots & * & x \end{pmatrix} \overset{\text{row-wise}}{=} f\begin{pmatrix} y & * & \cdots & * \\ * & y & * & \vdots \\ \vdots & * & \ddots & * \\ * & \cdots & * & y \end{pmatrix}$$

in part: identities cannot be witnessed by projections (unless $|A| = 1$)

Ex

- constant operations: $f(x \cdots x) = f(y \cdots y)$
- binary commutative: $\varphi_1 : f(x, y) = f(y, x)$

  $\varphi_2 : f(y, x) = f(x, y)$
- Mal'cev: $m(x, y, y) = m(y, y, x) = x$

  $m(x, x, x) = x$

  e.g. $m\begin{pmatrix} x & y & y \\ x & x & x \\ y & y & x \end{pmatrix} = m\begin{pmatrix} y & y & x \\ y & y & x \\ x & y & y \end{pmatrix}$

Def $\underline{A}$ τ-algebra. A Taylor term of $\underline{A}$ is τ-term $t(x_1 \ldots x_n)$, $n \geq 2$ s.t. $t^{\underline{A}}$ is Taylor op.

Recall: $\mathcal{C}, \mathcal{D}$ clones of operations. A clone homomorphism is mapping $\xi : \mathcal{C} \to \mathcal{D}$ that preserves arities, projections, and compositions.

equiv: ξ preserves identities

$\mathcal{P}roj$ = clone of projections on 2-element set

Thm (Taylor, '77) $\underline{A}$ idempotent. Then $\underline{A}$ has a Taylor term iff there is no clone homomorphism $Clo(\underline{A}) \to Proj$.

Proof

"$\Rightarrow$" Taylor identities cannot be witnessed by projections, clone homomorphisms preserve identities.

"$\Leftarrow$" assume $\nexists\, \xi : Clo(\underline{A}) \to Proj$.
by Cor. 6.5.14 $\exists$ pp-sentence $\underline{\Psi}$ that holds in $Clo(\underline{A})$ but not in $Proj$.

wlog. (by introducing new existentially quantified variables for subterms):

$$\Psi : \exists f_1 \ldots \exists f_r \, \varphi(f_1 \ldots f_r)$$

where $\varphi(f_1 \ldots f_r)$ is conjunction of atoms of the form

- $g(x_1 \ldots x_\ell) = g_0(g_1(x_1 \ldots x_\ell), \ldots, g_m(x_1 \ldots x_\ell))$

or

- $g(x_1 \ldots x_\ell) = pr^\ell_m(x_1 \ldots x_\ell)$,

$$g, g_0 \ldots g_m \in \{f_1 \ldots f_r\}$$

e.g. identity $f(x,y) = f(y,x)$ is equivalent to

$$\exists f_1, f_2, f_3 \big(f_1(x,y) = f_1(f_2(x,y), f_3(x,y)) \\ \wedge f_2(x,y) = pr^2_1(x,y) \\ \wedge f_3(x,y) = pr^2_2(x,y) \big)$$

for $f \in Clo^{\ell}(\underline{A})$, $g \in Clo^{(m)}(\underline{A})$ let

$$f * g(x_1 \dots x_{m\ell}) := f(g(x_1 \dots x_m), g(x_{m+1} \dots x_{2m}) \dots g(x_{(m-1)\ell+1} \dots x_{m\ell}))$$

- $f(x_1 \dots x_\ell) = f * g(\overbrace{x_1 \dots x_1}^{m\text{-times}}, x_2 \dots x_2 \dots x_\ell \dots x_\ell)$
- $g(x_1 \dots x_m) = f * g(\underbrace{\underbrace{x_1, x_2 \dots x_m}, \underbrace{x_1, x_2 \dots x_m} \dots \underbrace{x_1, x_2 \dots x_m}}_{\ell\text{-times}})$

by idempotency of $\underline{A}$

So far ...

- $Clo(\underline{A}) \models \underline{\Psi}$
- $Proj \not\models \underline{\Psi}$
- $\underline{\Psi}: \exists f_1 \dots \exists f_r \, \varphi(f_1 \cdots f_r)$

let k_i denote the arity of f_i, $i = 1..r$

$$k := \prod_{i=1}^{r} k_i$$

$$F := f_1 * (f_2 * (\cdots (f_{r-1} * f_r) \cdots)) \in Clo^{(k)}(\underline{A})$$

<u>Claim:</u> As in [illegible], every f_i, $i = 1..r$ is obtained by identifying variables in F:

$$f_i(x_1 \cdots x_{k_i}) = F\Big(\underbrace{(x_1^{k_{i+1} \cdots k_r} \cdots x_{k_i}^{k_{i+1} \cdots k_r}}_{k_i k_{i+1} \cdots k_r})^{k_1 \cdots k_{i-1}} \Big)$$

Proof of Claim:

r=2

$$f_1(x_1 \dots x_{k_1}) = f_1 * f_2\left(x_1^{k_2} \dots x_{k_1}^{k_2}\right)$$

$$f_2(x_1 \dots x_{k_2}) = f_1 * f_2\left((x_1 x_2 \dots x_{k_2})^{k_1}\right)$$

r≥3

$$F = f_1 * (f_2 * \dots \underbrace{f_i * (f_{i+1} * \dots * f_r) \dots)}_{=: F_i \in Cb^{(k_i \dots k_r)}(A)}$$

$$f_i(x_1 \dots x_{k_i}) \overset{r=2}{=} F_i\Big(\underbrace{x_1^{k_{i+1} \dots k_r}, x_2^{k_{i+1} \dots k_r} \dots x_{k_i}^{k_{i+1} \dots k_r}}_{=: \vec{x}}\Big)$$

$$f_{i-1} * F_i\left(\vec{x}^{k_{i-1}}\right) = f_{i-1}\left(F_i(\vec{x}) \dots F_i(\vec{x})\right) = F_i(\vec{x}) = f_i(x_1 \dots x_{k_i})$$

i-2 steps ⇝ $f_i(x_1 \dots x_{k_i}) = F\left(\vec{x}^{k_1 \dots k_{i-1}}\right)$

□

in other words: $f_i = F \circ \vec{p}^{\,i}$, where $\vec{p}^{\,i}$ is appropriate tuple of projections

let $n := k^2$ and $\lambda := F * F \in Clo^{(n)}(\underline{A})$

"every conjunct of Ψ can be written in the form $\lambda(\text{variables}) = \lambda(\text{variables})$"

- $f_i = \lambda \circ (\underbrace{\vec{p}^{\,i}, \vec{p}^{\,i} \dots \vec{p}^{\,i}}_{k\text{-times}}) =: \lambda \circ \vec{\vec{p}}^{\,i}$

- $f_i \circ (f_{i_1} \dots f_{i_{k_i}})(x_1 \dots x_j) =$

$$= (F \circ \vec{p}^{\,i}) \circ (F \circ \vec{p}^{\,i_1} \dots F \circ \vec{p}^{\,i_{k_i}})(x_1 \dots x_j) =$$

↑ picks k-many of $(F \circ \vec{p}^{\,i_1} \dots F \circ \vec{p}^{\,i_{k_i}})$

↑ picks k-many of $(x_1 \dots x_j)$

↑ picks k-many of $(x_1 \dots x_j)$

$$= F(F \circ \vec{p}^{\,i\vec{p}^{i}_1}, F \circ \vec{p}^{\,i\vec{p}^{i}_2} \dots F \circ p^{\,i\vec{p}^{i}_k})(x_1 \dots x_j) =$$

$$= \lambda \circ \vec{\supset}^{\,i, i_1 \dots i_{k_i}}(x_1 \dots x_j), \text{ where}$$

↑ picks n-many

$\vec{\supset}^{\,i, i_1 \dots i_{k_i}}$ is tuple of projections obtained from $\vec{p}^{\,i}$ by replacing every character $l \leq k_i$ by the tuple $\vec{p}^{\,i_l}$

⇒ every conjunct of the form

$$f_j(x_1 \ldots x_{k_j}) = f_i \circ (f_{i_1} \ldots f_{i_{k_i}})(x_1 \ldots x_{k_j})$$

can be written as

$$\lambda \circ \sqsupseteq^{j}(x_1 \ldots x_{k_j}) = \lambda \circ \sqsupseteq^{i, i_1 \ldots i_{k_i}}(x_1 \ldots x_{k_j})$$

- every conjunct of the form $f_j(x_1 \ldots x_{k_j}) = pr_e^{k_j}(x_1 \ldots x_{k_j})$ can be written as

$$\lambda \circ \sqsupseteq^{j}(x_1 \ldots x_{k_j}) = \lambda \circ (pr_e^{k_j} \ldots pr_e^{k_j})(x_1 \ldots x_{k_j})$$

Let ϑ be the conjunction of all these identities. Then $\mathrm{Clo}(\underline{A}) \models \exists \lambda\, \vartheta$.

Claim: If $\lambda \in \mathrm{Clo}^{(n)}(\underline{A})$ witnesses ϑ, then λ is a Taylor operation.

Proof of Claim:

w.t.s: $\forall \ell \leq n\ \exists \vec{v}, \vec{v}\,' \in \{x, y\},\ \vec{v}_\ell \neq \vec{v}_\ell\,'$: λ satisfies $\lambda(\vec{v}) = \lambda(\vec{v}\,')$.

i.e. $\lambda(* \cdots * \underset{\substack{\uparrow \\ \ell}}{x} * \cdots *) = \lambda(* \cdots * \underset{\substack{\uparrow \\ \ell}}{y} * \cdots *)$

assume $\exists \ell \, \forall \vec{v}, \vec{v}' \left(\lambda(\vec{v}) = \lambda(\vec{v}') \rightarrow \vec{v}_\ell = \vec{v}'_\ell \right)$ (*)

i.e.: same variable appears at the ℓ-th place of both sides.

write $\ell = (\ell_1 - 1) \cdot k + \ell_2$ with $\ell_1, \ell_2 \in \{1..k\}$

we must in fact have $\ell_1 = \ell_2 = \ell'$ since λ satisfies

$\lambda = \mp * \mp$

$$\vec{v}_\ell = x_{\ell_1}$$

$$\lambda(x_1 \cdots x_1, x_2 \cdots \mid x_2 \cdots x_k \cdots x_k) =$$

$$=$$

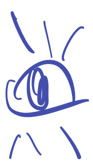
$$= \lambda(x_1 \cdots x_k, x_1 \cdots \mid x_k \cdots x_1 \cdots x_k)$$

$$\vec{v}'_\ell = x_{\ell_2}$$

But then $\Psi : \exists f_1 \dots \exists f_r \, \varphi(f_1 \dots f_r)$ is satisfied by the assignment

conj. of compositions and projections!

$$S : \begin{cases} f_i \longmapsto pr^{k_i}_{\vec{a}^i_\ell} , & i = 1 \dots r \\ pr^{k_j}_m \longmapsto pr^{k_j}_m \end{cases} ;$$



$$\vec{a}^i = (\overbrace{\vec{p}^i}^{\text{length } k} \; \underbrace{\vec{p}^i \dots \vec{p}^i}_{k\text{-times}}) , \quad \ell = (\ell' - 1) \cdot k + \ell'$$

$$\Rightarrow \vec{a}^i_\ell = \vec{p}^i_{\ell'}$$

Check for every conjunct

- $f_j = f_{i_0}(f_{i_1} \cdots f_{i_{k_i}})$ (w) $\lambda \circ \exists^j = \lambda \circ \exists^{i, i_1 \cdots i_{k_i}}$

  (f_j: conj. of φ; $\exists^j$: conj. of θ)

by $(*)$ $\Rightarrow$ $\exists^j_e = \exists^{i, i_1 \cdots i_{k_i}}_e$

hence $g(f_j) = pr^{k_j}_{\exists^j_e} = pr^{k_j}_{\exists^{i, i_1 \cdots i_{k_i}}_e}$.

$\vec{p}^{\,i,i_1\dots i_{k_i}}$ is tuple of projections obtained from $\vec{p}^{\,i}$ by replacing every character $m \le k_i$ by the tuple $\vec{p}^{\,i_m}$, $\ell = (\ell'-1)\cdot k + \ell'$

$$\Rightarrow \vec{p}^{\,i,i_1\dots i_{k_i}}_{\ell} = \left(\vec{p}^{\,i_{\vec{p}^{\,i}_{\ell'}}}\right)_{\ell'} \leftarrow \text{(ℓ'-th of the ℓ'-th tuple)}$$

hence $g(f_i)\circ\left(g(f_{i_1})\dots g(f_{i_{k_i}})\right) =$

$$= pr^{k_j}_{\vec{p}^{\,i}_{\ell'}} \circ \left(pr^{k_j}_{\vec{p}^{\,i_1}_{\ell'}} \dots pr^{k_j}_{\vec{p}^{\,i_{k_i}}_{\ell'}}\right) = pr^{k_j}_{\left(\vec{p}^{\,i_{\vec{p}^{\,i}_{\ell'}}}\right)_{\ell'}} \overset{*}{=} pr^{k_j}_{\vec{p}^{\,i,i_1\dots i_{k_i}}_{\ell}} = g(f_j)$$

• $f_j = pr^{k_j}_m \iff \lambda \circ \exists^j = \lambda \circ (pr^{k_j}_m \cdots pr^{k_j}_m)$

$\uparrow$ conj. of φ $\qquad\qquad$ $\uparrow$ conj of ϑ

by (*), $\exists^j_e = m$

then $g(f_j) = pr^{k_j}_{\exists^j_e} = pr^{k_j}_m = g(pr^{k_j}_m)$.

$\Rightarrow$ Ψ can be satisfied by projections. ↯ $\square$