Minor-preserving maps

& h1-identities

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&

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Disclaimer: (1) Watch the previous episodes!!!

specially: Zaneta
Moritz
Roman (x2)

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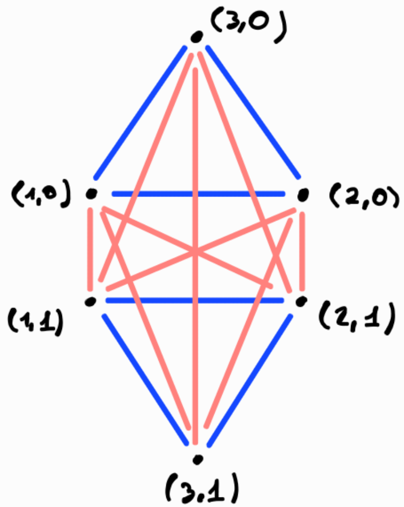
(2) In this talk: "Zaneta proved" := "Earlier, someone in this reading group presented the following"

Example: (Barto, Pinsker)



$$\mathbb{B} = (\underbrace{\{1,2,3\} \times \{0,1\}}_{:=B}; R, S)$$

$R = \{((a,i),(b,j)) \mid i = j \wedge a \neq b\}$

$S = \{((a,i),(b,j)) \mid i \neq j\}$

- $\mathbb{B}$ is a core ✓
- $\forall c \in B$, $(\mathbb{B}, c)$ pp-interprets $\mathbb{K}_3$ $\qquad \{x \in B \mid S(x,c)\} \cong \mathbb{K}_3$

  $\Rightarrow \mathbb{K}_3 \in HI(\mathbb{B})$
- $\mathbb{K}_3 \notin I(\mathbb{B})$ $\quad$ ($\mathrm{Pol}(\mathbb{B}) \not\to \mathrm{Proj}$)

$\alpha((a,i)) := (a, i-1)$ is a autom. of $\mathbb{B}$

$$s((a,i),(b,j),(c,k)) := \begin{cases} (c,k) & \text{if } i=j \\ (a,i) & \text{otw} \end{cases}, \quad s \in \mathrm{Pol}(\mathbb{B})$$

$$\Rightarrow \underbrace{\forall x,y: s(x,x,y) = y = s(y,\alpha(y),x)}_{\text{NON-TRIVIAL}}$$

In previous episodes

Corollary 6.5.16: Let $\mathbb{A}$ and $\mathbb{B}$ be rel. structures. Then $(1) \Rightarrow (2) \Rightarrow (3)$.

(1) $\mathbb{A} \in I(\mathbb{B})$;

(2) There is a clone homomorphism from $\mathrm{Pol}(\mathbb{B})$ to $\mathrm{Pol}(\mathbb{A})$;

(3) Every strong Mal'cev condition that holds in $\mathrm{Pol}(\mathbb{B})$ also holds in $\mathrm{Pol}(\mathbb{A})$.

If $\mathbb{A}$ and $\mathbb{B}$ are finite structures, then (1), (2), (3) are equivalent.

: if $\mathbb{A} \notin I(\mathbb{B})$, then $\exists\, \Sigma$ Mal'cev condition s.t. $\mathrm{Pol}(\mathbb{B}) \models \Sigma$ and $\mathrm{Pol}(\mathbb{A}) \not\models \Sigma$

? What about $\mathbb{A} \in HI(\mathbb{B})$?

We have seen: $\mathbb{A} \in HI(\mathbb{B}) \iff \exists\, \mathbf{A} \in \mathrm{Exp}\,\mathrm{Refl}(\mathbf{B})$ s.t. $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathbb{A})$.

a polym. alg. of $\mathbb{B}$.

- Do we have a correspondent notion of "morphism"?
- Which identities are preserved? (Birkhoff-style)

Def: A, B non-empty sets ; $f: A^n \to B$.

- ESSENTIALLY : A **minor** of f is any operation obtained from f by
  - identifying variables
  - permuting var.
  - adding new var.
- let $\pi : \{1, \dots, n\} \to \{1, \dots, k\}$, f_π denotes the k-ary operation

$$\underbrace{f_\pi}_{\text{a minor of } f}(x_1, \dots, x_k) := f(x_{\pi(1)}, \dots, x_{\pi(n)})$$

- A **minion** on (A, B) is a non-empty $\mathcal{M} \subseteq \bigcup_{k \geq 1} A^k \to B$ closed under taking minors.

: - Every operation clone is a minion ($A = B$).
- Minions are not required to contain projections.

Def: $\mathcal{M}_1, \mathcal{M}_2$ minions. A **minion homomorphism** is a map $\xi : \mathcal{M}_1 \to \mathcal{M}_2$ s.t.

a.k.a. minor-preserving maps

- ξ preserves arities;
- $\xi(f_\pi) = \xi(f)_\pi$, for any n-ary $f \in \mathcal{M}_1$ and $\pi : \{1, \dots, n\} \to \{1, \dots, k\}$

Example (minion hom. need not preserve projections)

Consider the clone $\mathrm{Pol}(\{0,1\}; \mathrm{NAE}) = \langle \neg \rangle$.

The map $\xi: f \mapsto \neg f$ is minor-preserving and does not preserve projections.

Prop. 6.7.4: $\mathcal{M}$ a minion; $\mathcal{P}$ a clone that only contains projections.

Then every minor-preserving map $\xi: \mathcal{M} \to \mathcal{P}$ preserves all projections in $\mathcal{M}$.

Proof: Suppose $\xi(\pi_i^k) \overset{(*)}{=} \pi_j^k$, for some $i \neq j$.

$$\Longrightarrow \pi_j^k \overset{(*)}{=} \xi(\pi_i^k) = \xi(\pi_i^k(\pi_i^k, \dots, \pi_i^k))$$

$$\overset{\xi \text{ m. pres.}}{=} \xi(\pi_i^k)(\pi_i^k, \dots, \pi_i^k)$$

$$\overset{(*)}{=} \pi_j^k(\pi_i^k, \dots, \pi_i^k) = \pi_i^k \quad ↯$$

Minor Conditions

Def: • A minor identity is an abstract expression of the form

$$\forall x_1,\dots,x_n, y_1,\dots,y_m \left(f(\underbrace{x_1,\dots,x_n}) = g(\underbrace{y_1,\dots,y_m})\right)$$

operation symbols

not necessarily distinct variables

• A minor condition is a finite set of minor identities.

Example:

✓ $m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx m(x,x,x)$	✗ $s(x,s(y,z)) \approx s(s(x,y),z)$
✓ $s(x,y) \approx s(y,x)$	✗ $f(x,\dots,x) \approx x$

: $\mathbf{A}, \mathbf{B}$ are τ-algebras. Then there exists a minion hom. $\xi : \mathrm{Minion}(\mathbf{B}) \longrightarrow \mathrm{Minion}(\mathbf{A})$ that maps

the smallest minion containing $\{f^{\mathbf{B}} \mid f \in \tau\}$

$$f^{\mathbf{B}} \mapsto f^{\mathbf{A}} \iff \forall f,g \in \tau : \quad f^{\mathbf{B}}(p_1,\dots,p_k) = g^{\mathbf{B}}(q_1,\dots,q_\ell)$$

$$\text{implies} \quad f^{\mathbf{A}}(p_1,\dots,p_k) = g^{\mathbf{A}}(q_1,\dots,q_\ell)$$

If ξ exists it must be surjective; we call ξ the natural minion homomorphism.

Thm ("Linear Birkhoff"; Barto, Opršal, Pinsker): Let $\mathbf{A}, \mathbf{B}$ be τ-alg. s.t. $\mathcal{A} := \mathrm{Minion}(\mathbf{A})$, $\mathcal{B} := \mathrm{Minion}(\mathbf{B})$ are operational clones. TFAE:

(1) The natural minion homomorphism from $\mathcal{B}$ to $\mathcal{A}$ exists;

(2) All minor identities that hold in $\mathbf{B}$ also hold in $\mathbf{A}$;

(3) $\mathbf{A} \in \mathrm{Refl}\, P^{\mathrm{fin}}(\mathbf{B})$ (if A and B are finite)

proof: (1) $\Leftrightarrow$ (2): ✓ (definitions; see also Thm 6.5.10)

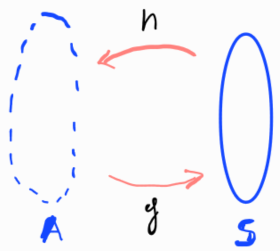
(2) $\Rightarrow$ (3): $\forall a \in A$, $\pi_a^A \in \underbrace{B^{B^A}}_{C}$ maps every tuple in B^A to its a-th coordinate

Consider $\mathbf{S} := \langle \{ \pi_a \mid a \in A \} \rangle \leq \mathbf{B}^{B^A}$.

Define $h: S \to A$, $h(f^{\mathbf{B}}(\pi_{a_1}^A, \dots, \pi_{a_n}^A)) := f^{\mathbf{A}}(a_1, \dots, a_n)$.

Define $g: A \to S$, $a \mapsto \pi_a^A$. We have $\forall\, a_1, \dots, a_n \in A$:

$$f^{\mathbf{A}}(a_1, \dots, a_n) = h(f^{\mathbf{B}}(g(a_1), \dots, g(a_n))).$$



Thus $\mathbf{A} \in \mathrm{Refl}\, P(\mathbf{B})$.

Thm ("Linear Birkhoff"; Barto, Opršal, Pinsker): Let $\mathbf{A}, \mathbf{B}$ be τ-alg. s.t. $\mathcal{A} := \mathrm{Minion}(\mathbf{A})$, $\mathcal{B} := \mathrm{Minion}(\mathbf{B})$ are operational clones. TFAE:

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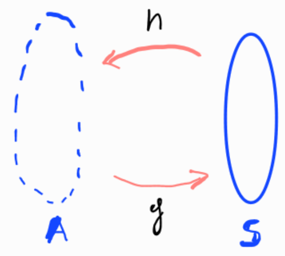


Thus $\mathbf{A} \in \mathrm{Refl}\,\mathrm{P}^{\mathrm{fin}}(\mathbf{B})$. (if A and B are finite $\Rightarrow$ B^A is finite)

Thm ("Linear Birkhoff"; Barto, Opršal, Pinsker): Let $\mathbf{A}, \mathbf{B}$ be τ-alg. s.t. $\mathcal{A} := \mathrm{Minion}(\mathbf{A})$, $\mathcal{B} := \mathrm{Minion}(\mathbf{B})$ are operational clones. TFAE:

(1) The natural minion homomorphism from $\mathcal{B}$ to $\mathcal{A}$ exists;

(2) All minor identities that hold in $\mathbf{B}$ also hold in $\mathbf{A}$;

(3) $\mathbf{A} \in \mathrm{Refl}\, P^{fin}(\mathbf{B})$ (if A and B are finite)

proof: (1) $\Leftrightarrow$ (2): ✓ (definitions; see also Thm 6.5.10) (2) $\Rightarrow$ (3): ✓

(3) $\Rightarrow$ (2) if $\mathbf{A} \in P$ ✓ (it follows from Birkhoff's thm (Thm 6.5.1))

Suppose $\mathbf{A}$ is a reflection of $\mathbf{B}$ via
$$\begin{tikzcd} A \arrow[r, bend right, "g"'] & B \arrow[l, bend right, "h"'] \end{tikzcd}$$
. Suppose $\mathbf{B} \models \Sigma$

where Σ is the identity $\forall x_1, \dots, x_n : f_1(x_{i_1}, \dots, x_{i_k}) = f_2(x_{j_1}, \dots, x_{j_l})$ $f_1, f_2 \in \tau$

Hence, $\forall a_1, \dots, a_n \in A$:
$$\begin{aligned} f_1^{\mathbf{A}}(a_{i_1}, \dots, a_{i_k}) &= h(f_1^{\mathbf{B}}(g(a_{i_1}), \dots, g(a_{i_k}))) \\ &= h(f_2^{\mathbf{B}}(g(a_{j_1}), \dots, g(a_{j_l}))) \\ &= f^{\mathbf{A}}(a_{j_1}, \dots, a_{j_l}). \end{aligned}$$

Thus $\mathbf{A} \models \Sigma$.

Corollary 6.7.12: Let $\mathbb{A}$ and $\mathbb{B}$ be rel. structures. Then $(1) \Rightarrow (2) \Rightarrow (3)$.

(1) $\mathbb{A} \in HI(\mathbb{B})$;

(2) There is a minion homomorphism from $Pol(\mathbb{B})$ to $Pol(\mathbb{A})$;

(3) Every minor condition that holds in $Pol(\mathbb{B})$ also holds in $Pol(\mathbb{A})$.

If $\mathbb{A}$ and $\mathbb{B}$ are finite structures, then (1), (2), (3) are equivalent.

proof: Let $\mathbf{B}$ be s.t. $Clo(\mathbf{B}) = Pol(\mathbb{B})$.

$(1) \Rightarrow (2)$: $\mathbb{A} \in HI(\mathbb{B}) \xRightarrow{\text{Thm } 6.4.3} \exists\, \mathbf{A} \in Exp\,Refl\,P^{fin}(\mathbf{B})$ s.t. $Clo(\mathbf{A}) = Pol(\mathbb{A})$

$\xRightarrow{\text{Thm } 6.7.8} \exists$ minion hom. from $Clo(\mathbf{B})$ to $Clo(\mathbf{A})$.

$(2) \Rightarrow (3)$: Let $\xi : Pol(\mathbb{B}) \to Pol(\mathbb{A})$ be a minion hom.; suppose $Pol(\mathbb{B}) \models \Sigma$. $(*)$

Take a conjunct of Σ: $f_\sigma \approx g_\rho$. We have:

$$\xi(f)_\sigma \overset{\text{m.h.}}{=} \xi(f_\sigma) \overset{(*)}{=} \xi(g_\rho) \overset{\text{m.h.}}{=} \xi(g)_\rho \implies Pol(\mathbb{A}) \models \Sigma.$$

Corollary 6.7.12: Let $\mathbb{A}$ and $\mathbb{B}$ be rel. structures. Then (1) ⇒ (2) ⇒ (3).

(1) $\mathbb{A} \in \mathrm{HI}(\mathbb{B})$;

(2) There is a minion homomorphism from $\mathrm{Pol}(\mathbb{B})$ to $\mathrm{Pol}(\mathbb{A})$;

(3) Every minor condition that holds in $\mathrm{Pol}(\mathbb{B})$ also holds in $\mathrm{Pol}(\mathbb{A})$.

If $\mathbb{A}$ and $\mathbb{B}$ are finite structures, then (1), (2), (3) are equivalent.

proof: Let $\mathbf{B}$ be s.t. $\mathrm{Clo}(\mathbf{B}) = \mathrm{Pol}(\mathbb{B})$.

(1) ⇒ (2): $\mathbb{A} \in \mathrm{HI}(\mathbb{B}) \xrightarrow{\text{Thm } 6.4.3} \exists\, \mathbf{A} \in \mathrm{Exp}\,\mathrm{Refl}\,\mathrm{P}^{\mathrm{fin}}(\mathbf{B})$ s.t. $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathbb{A})$

$\xrightarrow{\text{Thm } 6.7.8} \exists$ minion hom. from $\mathrm{Clo}(\mathbf{B})$ to $\mathrm{Clo}(\mathbf{A})$.

(2) ⇒ (3): ✓

(3) ⇒ (2): standard compactness argument (see Lemma 6.5.13)

Corollary 6.7.12: Let $\mathbb{A}$ and $\mathbb{B}$ be rel. structures. Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

(1) $\mathbb{A} \in HI(\mathbb{B})$;

(2) There is a minion homomorphism from $Pol(\mathbb{B})$ to $Pol(\mathbb{A})$;

(3) Every minor condition that holds in $Pol(\mathbb{B})$ also holds in $Pol(\mathbb{A})$.

If $\mathbb{A}$ and $\mathbb{B}$ are finite structures, then (1), (2), (3) are equivalent.

proof: Let $\mathbf{B}$ be s.t. $Clo(\mathbf{B}) = Pol(\mathbb{B})$.

(2) $\Rightarrow$ (1) (if $\mathbb{A}$ and $\mathbb{B}$ are finite): Let $\mathbf{A}$ have the same signature τ as $\mathbf{B}$ where $f \in \tau$ denotes $\xi(f^{\mathbf{B}})$, i.e., ξ is the natural minion hom. $Clo(\mathbf{B}) \to Clo(\mathbb{A})$.

$$\xrightarrow{\text{Thm. 6.7.8}} \mathbf{A} \in \text{Exp Refl } P^{fin}(\mathbf{B})$$

$$\Longrightarrow \exists\, \mathbf{A}' \in \text{Exp Refl } HSP^{fin}(\mathbf{B}) \text{ s.t. } Clo(\mathbf{A}') = Pol(\mathbb{A})$$

$$\xrightarrow{\text{Thm 6.4.3}} \mathbb{A} \in HI(\mathbb{B}).$$

The pp-constructability poset

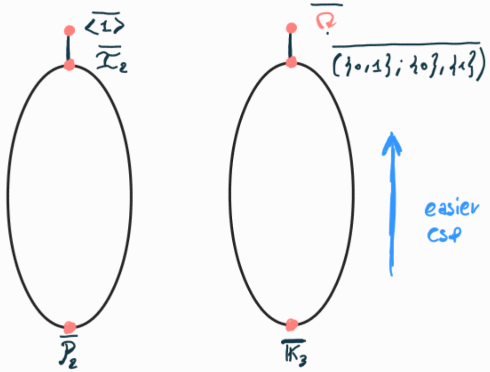
We write $\mathscr{C} \leq_m \mathscr{D}$ if there exists a minion hom. from $\mathscr{C}$ to $\mathscr{D}$.

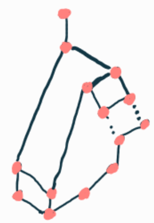
$\mathscr{C} \equiv_m \mathscr{D}$ iff $\mathscr{C} \leq_m \mathscr{D}$ and $\mathscr{D} \leq_m \mathscr{C}$. it is reflexive and transitive!

We denote by $\overline{\mathscr{C}}$ the $\equiv_m$-class of $\mathscr{C}$. Also, we write $\overline{\mathscr{C}} \leq_m \overline{\mathscr{D}}$ iff $\mathscr{C} \leq_m \mathscr{D}$.

$$\mathfrak{P}_{fin} := (\{\overline{\mathscr{C}} \mid \mathscr{C} \text{ is a clone over some finite set}\}; \leq_m)$$

$$\mathfrak{P}_{n} := (\{\overline{\mathscr{C}} \mid \mathscr{C} \text{ is a clone over } \{0,1,\dots,n-1\}\}; \leq_m)$$





Some results:

- Complete description of $\mathfrak{P}_2$ [Bodirsky, V.]
- Complete description of $\mathfrak{P}_{SD}$ (pol. clones of smooth digraphs) [Bodirsky, Starke, V.]
- Complete description of the lattice of clones of self-dual clones w.r.t. $\leq_m$ [Bodirsky, Zhuk, V]

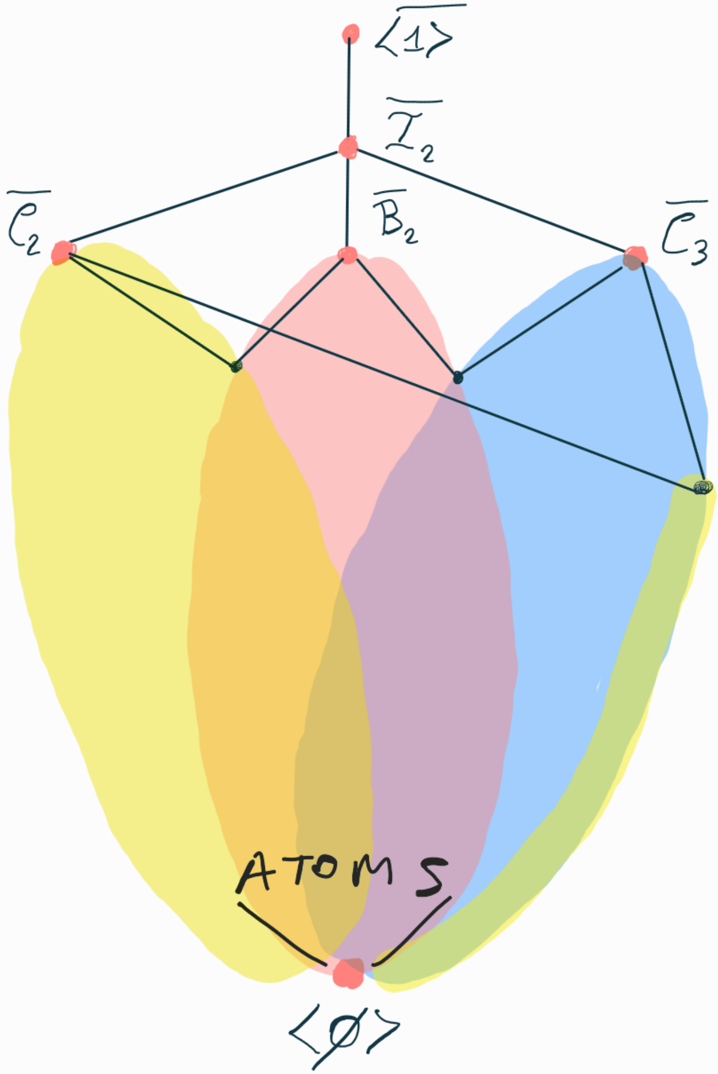
remarkably countably infinite!

An overview of $\mathcal{P}_3$

$\mathcal{C}_2 := \mathrm{Pol}(\quad)$

$\mathcal{C}_3 := \mathrm{Pol}(\quad)$

$\mathcal{B}_2 := \mathrm{Pol}(\{0,1\}; \{(0,1),(1,0),(1,1)\}, \{0\}, \{1\})$



: fully described [B V Z]

: potentially 2^ω elements

: work in progress [FIORAVANTI, ROSSI, V.]


Thank You!