§ 6.1.8 HELL-NEŠETŘIL THEOREM

HELL-NEŠETRIL THEOREM

Let B be a finite loopless undirected graph. Then,

EITHER B is bipartite (i.e. admits a homomorphism to K_2)

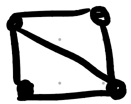
OR $K_3 \in HI(B)$.

structures homomorphically equivalent to structures pp-interpretable in B.

Note: B bipartite $\Rightarrow$ $CSP(B) \in P$

problem of whether an instance is bipartite, reduce to connected components and look for the (unique possible) bipartition.

$K_3 \in HI(B) \Rightarrow CSP(B)$ is NP-complete (COROLLARY 3.7.1)

A DIAMOND is the graph 

G is DIAMOND-FREE if it does not contain a copy of a diamond as a weak subgraph.

opposed to induced subgraph.

e.g. K_4 is NOT diamond-free

LEMMA 6.8.2 Let B be a finite loopless graph which is <u>NOT</u> bipartite. Then, $HI(B)$ contains a finite diamond-free core graph containing a triangle.

<u>Proof</u>:

WLOG:

(1) B is of minimal size among loopless non-bipartite graphs in $HI(B)$.

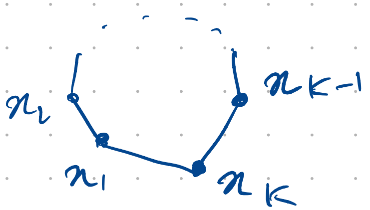
(otherwise, work with such minimal $B' \in HI(B)$. Then, $HI(B') \subseteq HI(B)$ by chap 3 stuff)

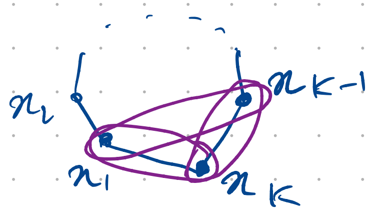
(2) B contains a triangle.

Recall: bipartite iff all cycles are even.

Say k is length of shortest odd cycle in B.

Take $(B; E^{k-2})$ for $E^{k-2} := \{(x_1, x_{k-1}) \mid \exists x_2 \dots x_{k-2}\ E(x_1,x_2) \wedge \dots \wedge E(x_{k-2},x_k)$





The new graph has same # of vertices as B and a triangle.

clearly still loopless.

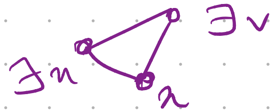
CLAIM 1: B is a core

otherwise Core(B) has fewer vertices and is still loopless non-bipartite ※ ①

CLAIM 2: Every vertex of B is contained in a triangle

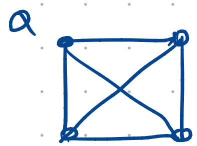
Otherwise look at subgraph of B induced on

$$A := \{x \in B \mid \exists u, v \, (E(x,u) \wedge E(x,v) \wedge E(u,v))\}$$



In this case $|A| < |B|$ and A is still loopless non-bipartite. ※ ①

CLAIM 3: B does <u>**NOT**</u> contain a copy of K_4

Otherwise there is some 

in B.

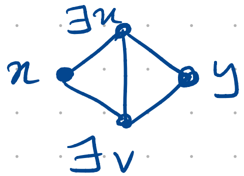
Look at subgraph of B induced on $A = \{x \in B \mid E(a,x)\}$

- $|A| < |B|$ since $a \notin A$
- $A \in HI(B)$ (by Prop 3.6.3 since B is a core $C(B) \subseteq HI(B)$ so $A \in IC(B) \subseteq HI(B)$)

expand

CLAIM 4: B is DIAMOND-FREE

Let $R(x,y) := \exists u,v\,(E(x,u) \wedge E(x,v) \wedge E(u,v) \wedge E(u,y) \wedge E(v,y))$



Note R is
- SYMMETRIC
- REFLEXIVE (every vertex is in a triangle by CLAIM3)

Let T be the transitive closure of R.

So, T is an equivalence relation on B.

Since B is finite, for some m,

$\delta_m(x,y) := \exists x_1 \dots x_{m-1}\,(R(x,x_1) \wedge \dots \wedge R(x_{m-1},y))$

defines T.

CLAIM 4.1: B/T is LOOPLESS

NTS $T \cap E = \emptyset$ (i.e. we are not identifying vs with an edge and making a loop)

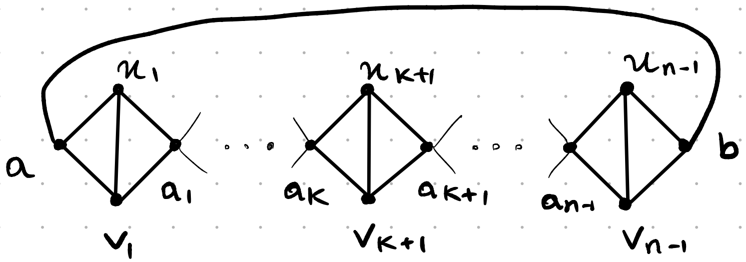
Proof by contradiction

Let $(a,b) \in T \cap E$ be s.t. $\delta_n(a,b)$ for n minimal.

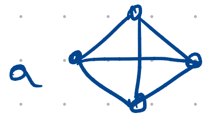
So $\exists a_0 \dots a_n$ s.t. $R(a_i, a_{i+1})$ in B

$a_0 = a$, $a_n = b$

We have the following picture:



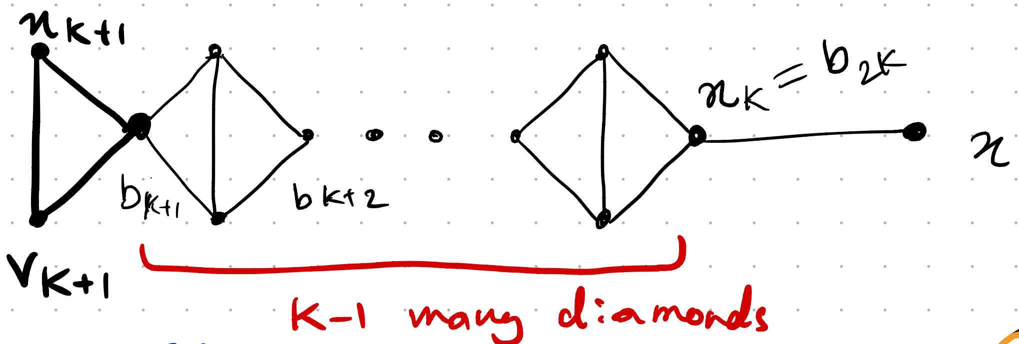
• **$n > 1$:** otherwise we would have

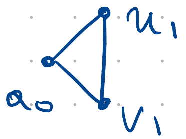


. So a copy of K_4 in B ※

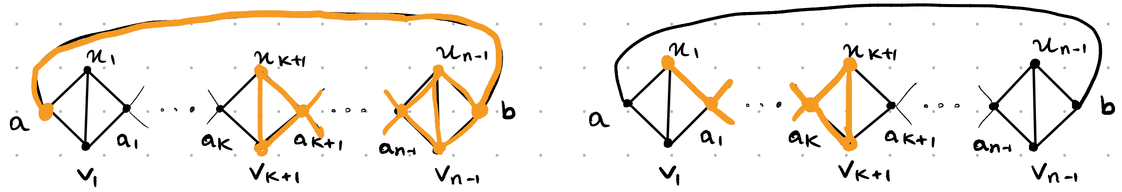
• **Suppose $n = 2K$.** Let

$$S = \{ x \in B \mid \exists x_1 x_K \; E(u_{K+1}, x_1) \wedge E(v_{K+1}, x_1) \wedge \delta_{K-1}(x_1, x_K) \wedge E(x_K, x) \}$$



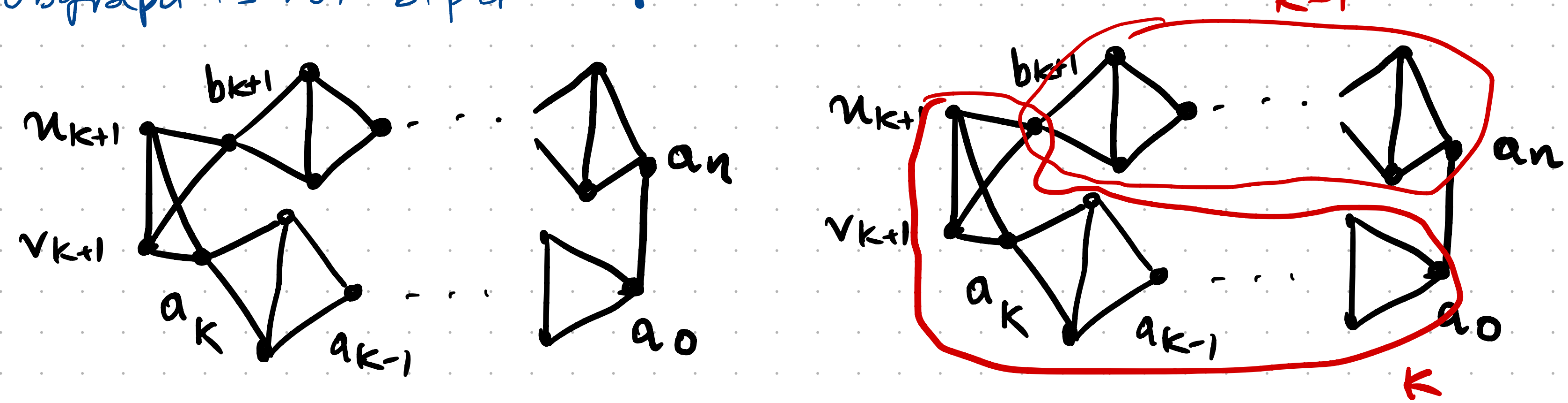
Now, 

is a triangle in S



So induced subgraph is not bipartite.

• and S:



this would give $\delta_{n-1}(a_0, a_n)$ contradicting minimality.

So $S \in H\,I(B)$ is loopless non-bipartite with $|S| < |B|$ ※.

• **For $n = 2K+1$,** a similar argument gives a contradiction

☑ CLAIM 4.1

So, $B/T \in HI(B)$ is loopless.

B has a triangle $\{abc\} \Rightarrow B/T$ has a triangle $\{a/T, b/T, c/T\}$.

Since B is of minimal size loopless non-bipartite, T is trivial.

So B is diamond-free ▨

LEMMA 6.8.4 Let B be a diamond-free loopless graph. Let $h:(K_3)^k \longrightarrow B$ be a homomorphism. Then, there is some $l \leq k$ s.t. $\mathrm{Im}\, h \cong (K_3)^l$.

<u>Proof</u>: Maybe later

LEMMA 6.8.5 Let B be a finite diamond-free loopless graph containing a triangle.
Then, B pp-interprets $(K_3)^k$ with parameters for some k.

Proof: Proof by CONTRADICTION.
We construct increasing $G_1 \subsetneq G_2 \subsetneq \dots$ subgraphs of B s.t.
$G_i \cong (K_3)^{k_i}$.
Since B is finite, we must eventually get a contradiction.

BC: G_1 is the triangle in B
IS: by ASSUMPTION, $G_i \cong (K_3)^{k_i}$ is NOT pp-definable in B with parameters.
So $\exists$ $f \in Pol(B)$ idempotent and $v_1 \dots v_k \in G_i$ s.t.

$$f(v_1 \dots v_k) \notin G_i \quad (\Delta)$$

So $f|_{G_i^k}$ gives a homomorphism $(K_3)^{k_i \cdot k} \longrightarrow B$.
By LEMMA 6.8.4, $G_{i+1} := f((G_i)^k)$ induces a copy of $(K_3)^{k_{i+1}}$.
• f is IDEMPOTENT + (Δ) $\Rightarrow$ $G_i \subsetneq G_{i+1}$. ∎

HELL-NEŠETRIL THEOREM

Let B be a finite loopless undirected graph. Then,

EITHER B is bipartite (i.e. admits a homomorphism to K_2)

OR $K_3 \in HI(B)$.

Proof: Say B is finite loopless non-bipartite.

By Lemma 6.8.2 $\exists A \in HI(B)$ a diamond-free core with a triangle.

By Lemma 6.8.5 $(K_3)^k \in IC(A) \subseteq HI(A)$

since A is a core

K_3 and $(K_3)^k$ are homom equivalent. So

$K_3 \in H(K_3)^k \subseteq HI(A) \subseteq HI(B)$. $\square$

EXTRA (If we had time)

Let $I = \{i_1, \dots, i_m\} \subseteq \{1, \dots, k\}$ $\quad i_1 < \dots < i_m$.

π^k_I is the function

$$(x_1, \dots, x_k) \longmapsto (x_{i_1}, \dots, x_{i_m})$$

For a map $h: A \to B$, **Ker h** is the equivalence relation on A where

$$\ker h(a, a') \text{ iff } h(a) = h(a').$$

LEMMA 6.8.4 Let B be a diamond-free loopless graph and $h: (K_3)^k \longrightarrow B$ be a homomorphism.

Then, there is $I \subseteq \{1, \dots, k\}$ s.t.

(a) h has the same kernel as π^k_I

(b) $\mathrm{Im}\, h \cong (K_3)^{|I|}$.

Proof: Let $I \subseteq \{1, \dots, k\}$ be maximal s.t. $\ker h \subseteq \ker \pi^k_I$.

I exists since $\ker \pi^k_\emptyset$ is the total relation on $(K_3)^k$.

(a) $\ker h = \ker \pi^k_I$: NTS $\forall j \in \{1, \dots, k\} \setminus I$ and $z_1 \dots z_k\, z_j' \in \{0, 1, 2\}$

$$h(z_1 \dots z_j \dots z_k) = h(z_1, \dots, z_j', \dots, z_k)$$

the tuples are related by $\ker \pi^k_I$

By MAXIMALITY of I $\exists\, x, y \in (K_3)^k$ s.t. $h(x) = h(y)$ $x_j \neq y_j$.

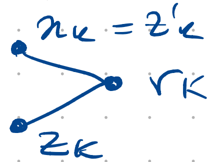
WLOG choose $z_j \neq x_j$ and $z'_j = x_j$ and assume $j = k$.

Any two vertices in $(K_3)^k$ have a common neighbour

e.g. for $k=3$ $(0,1,1)$, $(0,0,2)$ — $(1,2,0)$

- $r :=$ common nhour of x and $(z, z_k) = (z_1, \ldots z_{k-1}, z_k)$

  Since $x_k = z'_k$



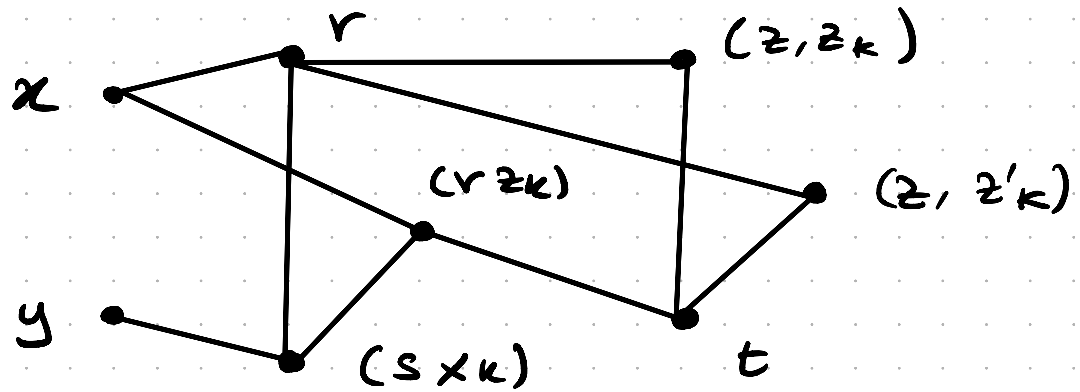
  $\Rightarrow$ r and (z, z'_k) are adjacent

- For $i < k$ let $s_i \notin \{r_i, y_i\}$.

  $x_k \notin \{r_k, y_k\} \Rightarrow (s\, x_k)$ is a common nhour of r and y

- (r, z_k) is a common nhour of x and $(s\, x_k)$
- For $i < k$ let $t_i \notin \{z_i, r_i\}$. $t_k \notin \{z_k, z'_k\}$ so

  t is a common nhour of (z, z_k), (z, z'_k) and (r, z_k)



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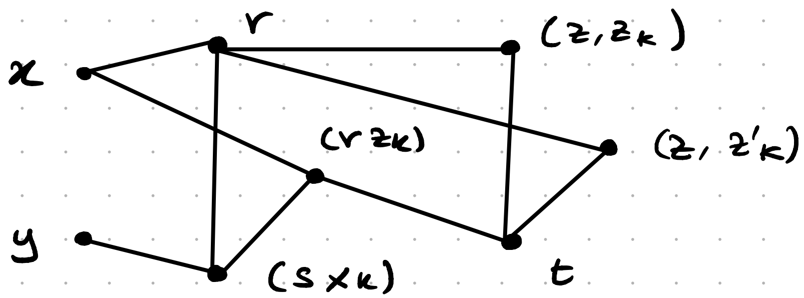
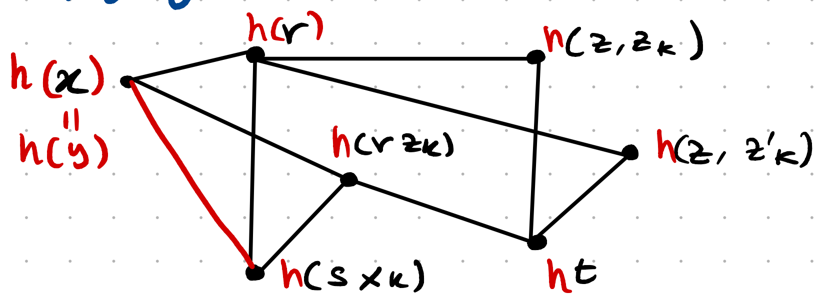


2 Applying h we know $h(x) = h(y)$



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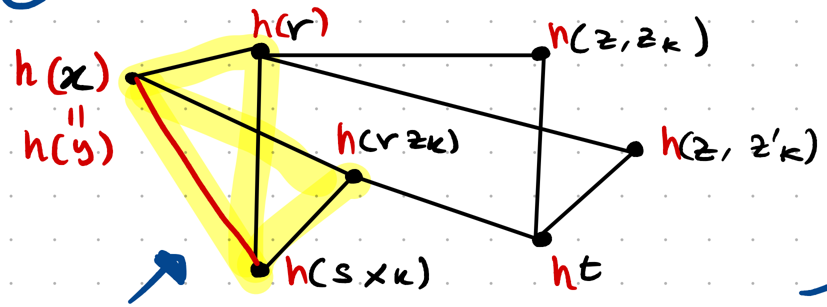
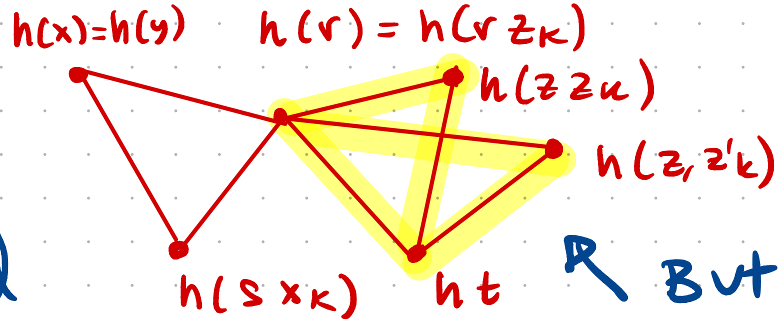


this is a DIAMOND!

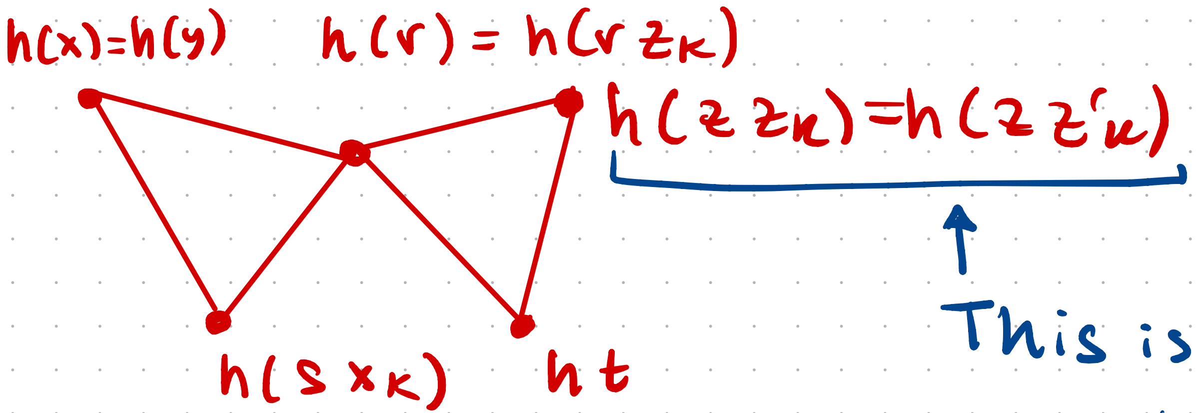
B is diamond free

4



But now this is a diamond!

5



This is what we wanted!

this completes the proof of $\ker h = \ker \pi_I^K$

(b) $\operatorname{Im} h \cong (K_3)^I$

we just prove $\pi_I^k \circ h^{-1}$ is an isomorphism $B \to K_3^I$

- well defined since they have same kernel
- to prove this is an isomorphism we prove similar tricks as the immediately preceding argument.

$\square$