MMSNP and examples

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19 July 2023

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MMSNP

Monotone Monadic SNP without Inequality

Definition

The MMSNP logic consists of ESO sentences of the form

$$\exists X_1, \ldots, X_s \, \forall x_1, \ldots, x_n \bigwedge_{i=1}^{m} \neg\big(\alpha_i \wedge \beta_i \wedge \varepsilon_i\big), \text{ where}$$

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- all existential relations $X_1, \ldots, X_s$ have arity 1 (monadic),
- every ε_i is empty (without inequality).

Complexity

Lemma

Every MMSNP is either in P or NP-complete if and only if every connected MMSNP is in P or NP-complete.

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Theorem ([FV'98])

For every connected MMSNP problem there exists a p-time equivalent finite CSP.

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Corollary ([Zhuk])

MMSNP has a "P vs NP-complete" dichotomy.

An equivalent definition

Let $\mathcal{F}$ be a finite family of finite structures over a signature $\tau \sqcup \sigma$, where $\sigma := \{\mathsf{M_1}(\cdot), \ldots, \mathsf{M_s}(\cdot)\}$ is a "set of colors".

$\mathsf{Forb_{hom}}(\mathcal{F})$

INSTANCE: A finite τ-structure $\mathbf{A}$.
QUESTION: Can we color each element of $\mathbf{A}$ with some color from σ such that for all $\mathbf{F}$ in $\mathcal{F}$ there is no homomorphism from $\mathbf{F}$ to the resulting σ-expansion $\mathbf{A}^\sigma$?

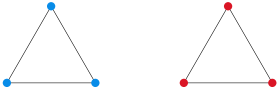
Lemma

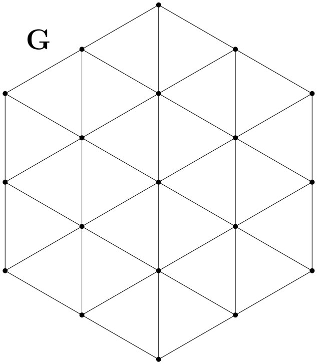
For every MMSNP sentence Φ there is a family $\mathcal{F}$ such that $\mathbf{A} \models \Phi$ if and only if $\mathbf{A} \in \mathsf{Forb_{hom}}(\mathcal{F})$ for every finite structure $\mathbf{A}$.

Example

No Monochromatic Triangle

Given a graph $\mathbf{G}$, color its vertices with 2 colors so that the result omits the two following subgraphs.





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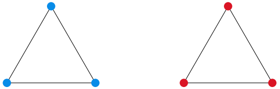
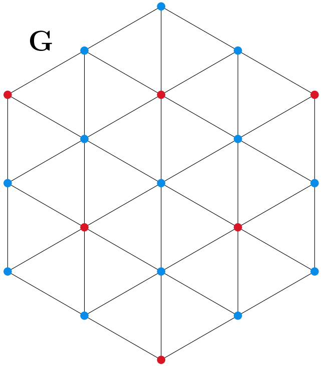
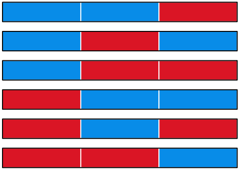


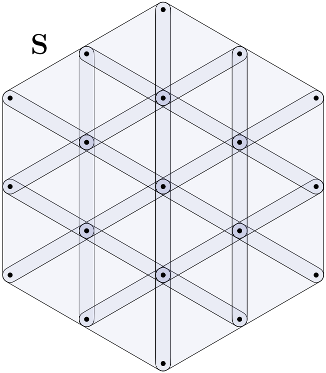


MMSNP reduces to finite CSP

- Replace every triangle of the input graph $\mathbf{G}$ with a relational triple.
- The original graph $\mathbf{G}$ is a YES instance iff the resulting structure $\mathbf{S}$ maps to $\mathbf{T}$, where $\mathbf{T}$ is as follows.



$\mathbf{T}$



The other direction

Naive approach

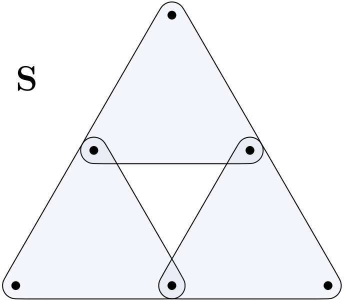
- Replace every relational triple of $\mathbf{S}$ with a triangle.
- Check if the resulting graph $\mathbf{G}$ satisfies the MMSNP sentence.

Obstacle

What to do when $\mathbf{S}$ contains implicit triangles?

Reason

$\mathbf{S}$ has cycles of length ≤ 3.

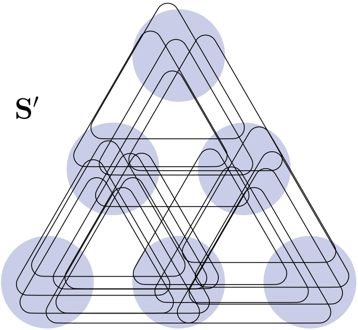


Solution

Lemma [Erdős'59]

For given $\mathbf{S}$, $\mathbf{T}$, and $\ell > 0$ there is $\mathbf{S}'$ such that

- $\mathbf{S} \to \mathbf{T}$ iff $\mathbf{S}' \to \mathbf{T}$;
- $\mathbf{S}'$ has no cycles of length less than ℓ.

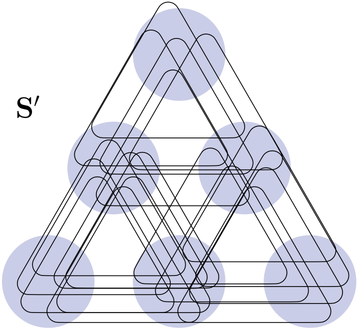


Solution

Construction of $\mathbf{S}'$

- Replace every vertex of $\mathbf{S}$ with a "bag" of size N.
- For every relational triple of $\mathbf{S}$, uniformly randomly distribute $\mathsf{N}^{1+\epsilon}$ triples on the corresponding three bags in $\mathbf{S}'$.
- Remove relational tuples until there are no cycles of length $< \ell$.

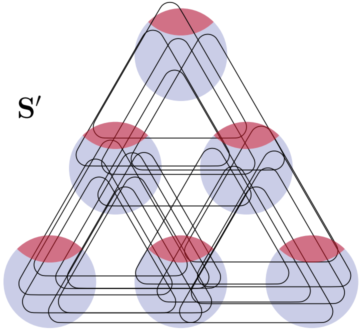
$\mathbf{S}'$



Solution

Proof

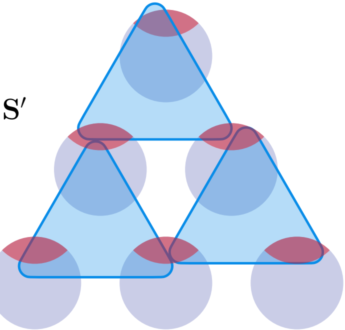
- By construction, $\mathbf{S}' \to \mathbf{S}$.
- The number of cycles of length $< \ell$ is small: $O(N^{\epsilon\ell})$, so we need to remove $O(N)$ tuples to get rid of them.
- If $\mathbf{S}' \to \mathbf{T}$, then each "bag" of size N contains at least $\frac{N}{|T|}$ vertices that are mapped to the same vertex in $\mathbf{T}$.



Solution

Proof

- As tuples are distributed uniformly randomly, even after removing $O(N)$ of them, there still are tuples induced on these smaller "subbags".
- If we map every vertex of $\mathbf{S}$ where the corresponding "subbag" is mapped, then it will be a homomorphism.



Why is it important?

Lemma [BMM'2021]

For every Φ in MMSNP there is an ω-categorical structure $\mathbf{C}_\Phi^\tau$ such that

- for every finite $\mathbf{A}$, we have $\mathbf{A} \models \Phi$ iff $\mathbf{A} \to \mathbf{C}_\Phi^\tau$,
- there is a one-to-one correspondence between 1-orbits of $\mathbf{C}_\Phi^\tau$ and the elements of $\mathbf{T}$,
- consequently, every canonical polymorphism of $\mathbf{C}_\Phi$ induces a polymorphism of $\mathbf{T}$.

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- consequently, every canonical polymorphism of $\mathbf{C}_\Phi$ induces a polymorphism of $\mathbf{T}$.

Theorem ([BMM'2021])

If $\mathbf{C}_\Phi$ has a non-trivial canonical polymorphism, then so does $\mathbf{T}$. Then, CSP$(\mathbf{T})$ is in P [Zhuk] as well as Φ.
If all canonical polymorphisms of $\mathbf{C}_\Phi$ are trivial, then so are all the polymorphisms. Then, CSP$(\mathbf{C}_\Phi)$ is NP-complete by [BOP'2018].

Horn-SAT

The problem

Definition

A propositional formula in conjunctive normal form is called Horn if each clause is a Horn clause, i.e., has at most one positive literal.

Example

$(\neg x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_4 \vee x_5) \wedge (\neg x_6 \vee \neg x_7 \vee \neg x_8) \wedge (x_9)$

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Horn-SAT

INSTANCE: A propositional Horn formula.
QUESTION: Is there a Boolean assignment for the variables such that in each clause at least one literal is true?

Complexity

Linear in the size of the input [DG'84]

An equivalent CSP

- Denote $C_0 := \{0\}$ and $C_1 := \{1\}$.
- For $a_1, \ldots, a_k \in \{0, 1\}$, $S_{a_1 \ldots a_k} := \{0, 1\}^k \setminus (a_1, \ldots, a_k)$.

Proposition

All $S_{11\ldots10}$ and $S_{11\ldots1}$ are pp-definable in C_0, C_1, and $H := S_{110}$.

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- $S_0(x) \leftrightarrow C_1(x)$. $S_{10}(x_1, x_2) \leftrightarrow H(x_1, x_1, x_2)$.

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- $S_{11\ldots10}(x_1, \ldots, x_m) \leftrightarrow \exists y_3, \ldots, y_{m-1}\ H(x_1, x_2, y_3)$
  $\wedge H(y_3, x_3, y_4) \wedge \cdots \wedge H(y_{m-1}, x_{m-1}, x_m)$.

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  $\wedge H(y_3, x_3, y_4) \wedge \cdots \wedge H(y_{m-1}, x_{m-1}, x_m)$.
- $S_{11\ldots1}(x_1, \ldots, x_m) \leftrightarrow \exists y\ S_{11\ldots10}(x_1, \ldots, x_m, y) \wedge C_0(y)$.

An equivalent CSP

- Denote $C_0 := \{0\}$ and $C_1 := \{1\}$.
- For $a_1, \dots, a_k \in \{0,1\}$, $S_{a_1 \dots a_k} := \{0,1\}^k \setminus (a_1, \dots, a_k)$.

Proposition

All $S_{11\dots10}$ and $S_{11\dots1}$ are pp-definable in C_0, C_1, and $H := S_{110}$.

Corollary

Let $\mathbf{A} := (\{0,1\}; C_0, C_1, H)$. *Then Horn-SAT* $\sim_p$ *CSP*$(\mathbf{A})$.

Algebraic properties of Horn-SAT

Definition

A mapping $\mathsf{min}\colon \{0,1\}^2 \to \{0,1\}$ is called binary minimum if $\mathsf{min}(0,0) = \mathsf{min}(0,1) = \mathsf{min}(1,0) = 0$ and $\mathsf{min}(1,1) = 1$.

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$2 \Rightarrow 1$

First, prove for any mapping f and any relations R_1, R_2: if R_2 is pp-definable in R_1 and f preserves R_1, then f preserves R_2. Then check that min preserves C_0, C_1, H.

$1 \Rightarrow 2$

For $[n] := \{1, \dots, n\}$ and $\mathbf{a} \in \{0,1\}^n$, put $\chi_{\mathbf{a}} := \{i \in [n] \colon a_i = 1\}$ and $\chi_R := \{\chi_{\mathbf{a}} \colon \mathbf{a} \in R\}$.

Case when $[n] \in \chi_R$

For $X \subseteq [n]$, define its "closure"

$$\mathrm{cl}(X) := \bigcap_{Y \colon X \subseteq Y, Y \in \chi_R} Y$$

It is the meet of all relational tuples above X. It is non-empty because $[n] \in \chi_R$.

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Claim: $X \in \chi_R$ iff $X = \mathrm{cl}(X)$

$\Rightarrow$: obvious

$\Leftarrow$: $Y_1, Y_2 \in \chi_R$ implies $Y_1 \cap Y_2 \in \chi_R$ as R is preserved by min.

$1 \Rightarrow 2$

Pp-definition

$$R(a_1, \ldots, a_n) \leftrightarrow \bigwedge_{X\colon X=\{i_1,\ldots,i_k\}\subseteq[n]} \; \bigwedge_{j\colon j\in cl(X)} S_{11\ldots10}(a_{i_1}, \ldots, a_{i_k}, a_j)$$

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Correctness proof

$\Rightarrow$: Consider $X \subseteq [n]$ s.t. $a_{i_1} = \cdots = a_{i_k} = 1$. Then, $X \subseteq \chi_{\mathbf{a}}$ and also $cl(X) \subseteq cl(\chi_{\mathbf{a}})$. By the claim, $cl(\chi_{\mathbf{a}}) = \chi_{\mathbf{a}}$. Thus, $a_j = 1$.
$\Leftarrow$: If $\mathbf{a} \notin R$, then $\exists j \in cl(\chi_{\mathbf{a}}) \setminus \chi_{\mathbf{a}}$. So, for $X = \chi_{\mathbf{a}}$, the corresponding atomic formula on the right hand side will not hold.

$1 \Rightarrow 2$

Case when $[n] \notin R$

We have to define the "closure" differently. For all $X \in [n]$ s.t. there is $Y \in \chi_R$ containing X, we put

$$\mathrm{cl}(X) := \bigcap_{Y:\, X \subseteq Y, Y \in \chi_R} Y$$

and $\mathrm{cl}(X) := [n]$ otherwise. Note that the claim still holds.

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Pp-definition

For $\mathbf{a} = (a_1, \ldots, a_n)$,

$$R(\mathbf{a}) \leftrightarrow S_{1\ldots1}(\mathbf{a}) \wedge \bigwedge_{X:\, X=\{i_1,\ldots,i_k\} \subseteq [n]} \bigwedge_{j:\, j \in \mathrm{cl}(X)} S_{11\ldots10}(a_{i_1}, \ldots, a_{i_k}, a_j)$$

CSPs over the integers

$(\mathbb{Z}, \mathrm{Succ})$

Definition

$\mathrm{Succ} = \{(x, y) : x + 1 = y\}$.

$$\cdots \longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots$$

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Observation

$(\mathbb{Z}, \mathrm{Succ})$ is not ω-categorical as there are infinitely many 2-orbits: $O_n = \{(x, y) : x + n = y\}$ for $n \in \mathbb{Z}$.

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Proposition

There is a sentence Φ in monotone connected SNP that describes $\mathrm{CSP}(\mathbb{Z}, \mathrm{Succ})$.

Proof

Φ existentially quantifies two binary relations Tc and Eq s.t.

- Tc contains Succ: $\mathrm{Succ}(x, y) \rightarrow \mathrm{Tc}(x, y)$

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- Eq is an equivalence relation: reflexive, symmetric: $\mathrm{Eq}(x, y) \rightarrow \mathrm{Eq}(y, x)$, and transitive
- The classes of Eq contain vertices on the "same level": $\mathrm{Eq}(v, w) \wedge \mathrm{Succ}(v, x) \wedge \mathrm{Succ}(w, y) \rightarrow \mathrm{Eq}(x, y)$, and $\mathrm{Eq}(v, w) \wedge \mathrm{Succ}(x, v) \wedge \mathrm{Succ}(y, w) \rightarrow \mathrm{Eq}(x, y)$

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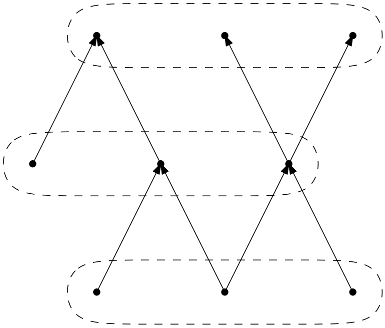
- Elements on the "same level" agree wrt Tc: $\mathrm{Tc}(x, y) \wedge \mathrm{Eq}(y, z) \rightarrow \mathrm{Tc}(x, z)$

In particular, we forbid the same Eq-class to contain a Tc pair.

Proof

- If there exists a homomorphism $h\colon \mathbf{G} \to (\mathbb{Z}, \mathsf{Succ})$, then interpret Eq as: $\mathsf{Eq}(x,y) \leftrightarrow \big(h(x) = h(y)\big)$, and put Tc to be minimal by inclusion.
- If there is a valid interpretation of Eq and Tc in $\mathbf{G}$, then $\mathbf{G}/\mathsf{Eq}$ is the disjoint union of finitely many directed paths without loops, then $\mathbf{G}/\mathsf{Eq} \to (\mathbb{Z}, \mathsf{Succ})$, and so does $\mathbf{G}$.



$(\mathbf{G}, \mathsf{Eq})$

$\mathbf{G}/\mathsf{Eq}$

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