Chapter 2.3: Fraïssé Amalgamation

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2.3.1. The age of a structure

Def • $\mathbb{B}$ structure; $\mathrm{Age}(\mathbb{B}) := \{ \mathbb{A} \text{ finite} \mid \mathbb{A} \hookrightarrow \mathbb{B} \}$

• $\mathcal{C}$ class of structures; $\mathcal{C}$ has the joint embedding property (JEP) if $\forall \mathbb{B}_1, \mathbb{B}_2 \in \mathcal{C}\ \exists \mathbb{C} \in \mathcal{C}$ such that

$$\mathbb{B}_1 \hookrightarrow \mathbb{C} \quad \text{and} \quad \mathbb{B}_2 \hookrightarrow \mathbb{C}.$$

Prop. 2.3.1. $\mathcal{C}$ class of finite τ-structures. TFAE:

1) $\mathcal{C} = \mathrm{Age}(\mathbb{B})$ for countably infinite $\mathbb{B}$
2) $\mathcal{C}$ closed under isomorphisms, substructures, contains countably many isomorphism types, has the JEP. ■

Example $\mathcal{C} := \{ \bullet\!-\!\bullet\!-\!\bullet \text{-free graphs} \}$

👁: $\mathrm{Age}(\mathbb{B}) = \mathrm{Age}(\mathbb{B}') \Rightarrow \mathrm{CSP}(\mathbb{B}) = \mathrm{CSP}(\mathbb{B}')$

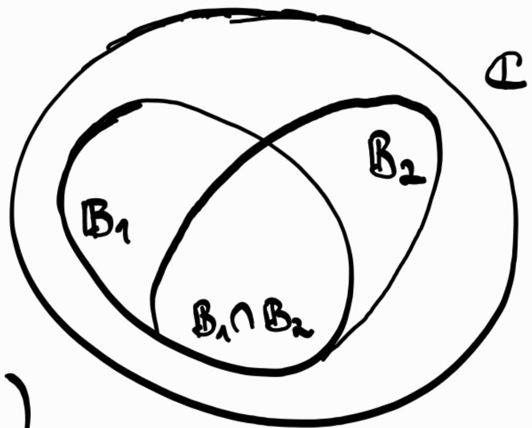
2.3.2. The amalgamation property

Def $\mathcal{C}$ class of structures closed under isomorphism

- $\mathcal{C}$ has the amalgamation property (AP) if

  $\forall\, \mathbb{B}_1, \mathbb{B}_2 \in \mathcal{C}\ \exists\, \mathbb{C} \in \mathcal{C}$ and $f_i : \mathbb{B}_i \hookrightarrow \mathbb{C}$ $(i \in \{1,2\})$ s.t.

$$f_1|_{B_1 \cap B_2} = f_2|_{B_1 \cap B_2}$$



- free AP: $\mathbb{C} = \mathbb{B}_1 \cup \mathbb{B}_2$
- strong AP: $f_1(B_1) \cap f_2(B_2) = f_1(B_1 \cap B_2)$

- $\mathcal{C}$ **amalgamation class** $:\Longleftrightarrow$ closed under isomorphisms, substructures, contains countably many isomorphism types, and has the AP.

: for relational signatures: AP $\Rightarrow$ JEP

<u>Example 2.3.3</u> $\mathcal{C} := \{$finite $\{<\}$-structures where $<$ denotes linear order$\}$

- $\mathcal{C}$ is an amalgamation class: given $B_1, B_2 \in \mathcal{C}$ the **free amalgam** $\mathbb{C} := B_1 \cup B_2$ is an acyclic directed graph; any linear extension of $\mathbb{C}$ is an **amalgam** (rest easy)

<u>Example 2.3.4</u> $\mathcal{C} := \{K_n$-free graphs$\}$ for $n \geq 3$ is a free amalgamation class

Example 2.3.5. $\mathcal{C} := \{$finite $\{E\}$-structures where E denotes an equivalence relation with at most k classes$\}$

- $\mathcal{C}$ has the strong AP but not the free AP

Example 2.3.6. $\mathcal{C} := \{$finite $\{E\}$-structures where E denotes an equivalence relation with classes of size $\leq k\}$

- $\mathcal{C}$ has the AP but not the strong AP

Def $\mathbb{B}$ homogeneous $:\Longleftrightarrow$ every isomorphism between finite substr. of $\mathbb{B}$ can be extended to an automorphism of $\mathbb{B}$

Prop. 2.3.7 $\mathbb{D}$ homogeneous $\Rightarrow$ $\mathrm{Age}(\mathbb{D})$ has the AP

Proof: • given $\mathbb{B}_1, \mathbb{B}_2 \in \mathrm{Age}(\mathbb{D})$ let $g_i : \mathbb{B}_i \hookrightarrow \mathbb{D}$ be arbitrary.

$\Rightarrow \quad \mathbb{D}[g_1(B_1 \cap B_2)] \cong \mathbb{D}[g_2(B_1 \cap B_2)]$

$\overset{\text{homogeneity}}{\Longrightarrow} \quad \exists f \in \mathrm{Aut}(\mathbb{D}) : \mathbb{D}[f \circ g_1(B_1 \cap B_2)] = \mathbb{D}[g_2(B_1 \cap B_2)]$

$\Rightarrow \quad \mathbb{C} := \mathbb{D}[g_1(B_1) \cup f^{-1} \circ g_2(B_2)]$ is an amalgam

<u>Thm. 2.3.8</u> (Fraïssé) $\mathcal{C}$ class of finite τ-str., τ countable
Then TFAE:

1) $\mathcal{C}$ is an amalgamation class
2.) $\exists$ up to isomorphism unique countable homogeneous str. $\mathbb{B}$ st $\mathcal{C} = \mathrm{Age}(\mathbb{B})$ (called Fraïssé limit)

<u>Examples</u> 2.3.9, 10, 11: • finite loopless undirected graphs
• finite K_n-free undirected graphs
• partial orders

2.3.3 Forbidden substructures

Def $\mathcal{F}$ set of finite τ-structures;

$$\mathrm{Forb}_e(\mathcal{F}) := \{\mathbb{A} \text{ finite } \tau\text{-str.} \mid \mathbb{F} \not\hookrightarrow \mathbb{A} \ \forall \mathbb{F} \in \mathcal{F}\}$$

Example 2.3.12 tournament :$\Leftrightarrow$ orientation of a clique

$\mathcal{F}$ set of finite tournaments $\Rightarrow$ $\mathrm{Forb}_e(\mathcal{F} \cup \{\bullet\circlearrowleft\})$ is an amalgamation class

Henson 1972: $\exists$ tournaments $T_1, T_2, \ldots$ s.t. $T_i \not\hookrightarrow T_j \ \forall i \neq j$

$\Rightarrow$ all distinct $\mathcal{F} \subseteq \{T_1, T_2, \ldots\}$ specify distinct Fraïssé limits $\mathbb{F}_{\mathcal{F}}$ with. $\mathrm{Age}(\mathbb{F}_{\mathcal{F}}) = \mathrm{Forb}_e(\mathcal{F}, \circlearrowleft)$

$\Rightarrow$ some homogeneous digraphs have undecidable CSP

Def. class $\mathcal{C}$ of finite τ-structures is finitely bounded if $\exists$ finite $\mathcal{F}$ st. $\mathcal{C} = \mathrm{Forb}_e(\mathcal{F})$; $\mathcal{F}$ is a set of bounds

Lemma 2.3.14. τ relational signature, $\mathcal{C}$ a class of finite τ-structures. TFAE:

1) $\mathcal{C}$ is finitely bounded
2) $\mathcal{C} = \mathrm{Mod}_{fin}(\Phi)$ for some universal τ-sentence Φ $\blacksquare$

most classes mentioned so far were finitely bounded

Proposition 2.3.15 $\mathbb{B}$ reduct of a finitely bounded structure, then $\mathrm{CSP}(\mathbb{B})$ is in NP

Proof. $\mathrm{CSP}(\mathbb{B})$ expressible in SNP $\blacksquare$

Chapter 13: CSPs of reducts of finitely bounded str. do not have a dichotomy $\rightsquigarrow$ add homogeneity as req.

Prop 2.3.16 $\mathbb{B}$ first-order definable in a reduct of a finitely bounded homogeneous structure $\mathbb{A}$ $\Rightarrow$ $\mathbb{B}$ reduct of some finitely bounded homogeneous str. $\mathbb{A}'$

Proof: $Age(\mathbb{A}) = Mod_{fin}(\Phi)$ for a universal sentence Φ

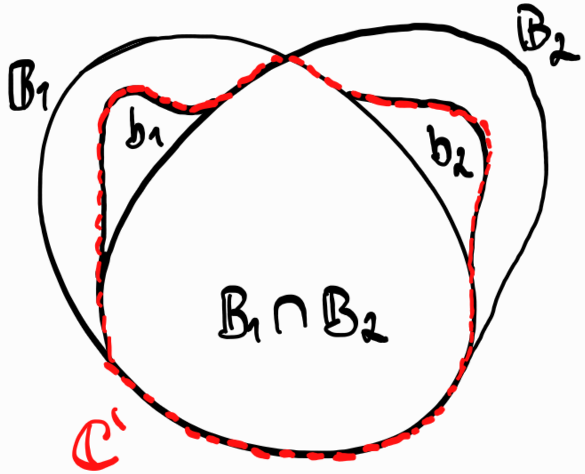
- $\mathbb{A}$ has quantifier elimination $\Rightarrow$ rel. of $\mathbb{B}$ have quantifier-free def. in $\mathbb{A}$
- add these to Φ $\rightsquigarrow$ a new sentence for $Age(\mathbb{A}')$ ■

Conjecture 2.1 $\mathbb{B}$ reduct of a finitely bounded hom. str. then $CSP(\mathbb{B})$ is in P or NP-complete.

2.3.4. One-point amalgamation

Prop. 2.3.17 $\mathcal{C}$ class of finite τ-str. closed under isomorphisms and substructures. TFAE:

1) $\mathcal{C}$ has the AP
2) $\mathcal{C}$ has the one-point AP, i.e. restricted to B_1, B_2 s.t. $|B_1| = |B_2| = |B_1 \cap B_2| + 1$

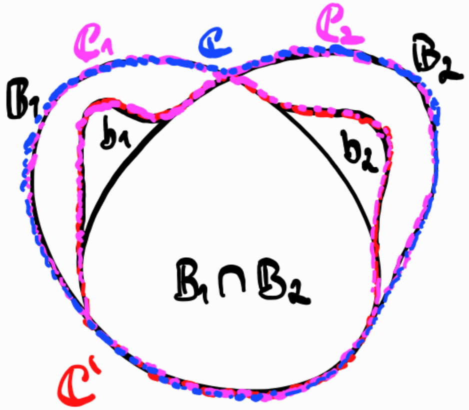


Proof (1) $\Rightarrow$ (2): Trivial

(2) $\Rightarrow$ (1): Induction on $|B_1 \setminus B_2| + |B_2 \setminus B_1|$

- select arbitrary $b_1 \in B_1 \setminus B_2$, $b_2 \in B_2 \setminus B_1$
- by (2), $B_1[\{b_1\} \cup (B_1 \cap B_2)]$, $B_2[\{b_2\} \cup (B_1 \cap B_2)]$ have amalgam $C' \in \mathcal{C}$

- by IH, $\mathbb{B}_1, \mathbb{C}'$ and $\mathbb{B}_2, \mathbb{C}'$ have amalgams $\mathbb{C}_1, \mathbb{C}_2 \in \mathcal{C}$
- by IH, $\mathbb{C}_1$ and $\mathbb{C}_2$ have an amalgam $\mathbb{C} \in \mathcal{C}$ ■



<u>Prop. 2.3.18</u> $\mathcal{C}$ class of finite τ-str. closed isomorphisms and substructures. TFAE:

1) $\mathcal{C}$ has the strong AP
2) $\mathcal{C}$ has the one-point strong AP, i.e. restricted to $\mathbb{B}_1, \mathbb{B}_2$ s.t. $|\mathbb{B}_1| = |\mathbb{B}_2| = |B_1 \cap B_2| + 1$ ■

2.3.5 Deciding the AP.

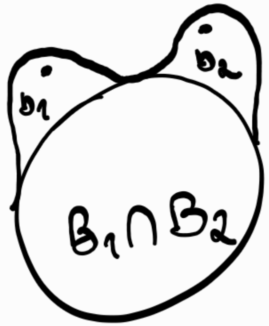
<u>Braunfeld 2015</u>: JEP undecidable already for finitely bounded classes of undirected graphs

Prop. 2.3.19 τ finite binary rel. signature, $\mathcal{F}$ finite set of finite τ-structures, $\mathcal{C} := \mathrm{Forb}_e(\mathcal{F})$. Then TFAE:

1) $\mathcal{C}$ has the AP
2.) $\mathcal{C}$ has the 1-point AP restricted to pairs $\mathbb{B}_1, \mathbb{B}_2$ of size $\leq (m-2)\cdot \ell$ where:

i) $\ell := |\{\mathbb{A} \in \mathcal{C} \mid A = \{1,2\}\}|$
ii) $m := \max \{3\} \cup \{|F| \mid F \in \mathcal{F}\}$

Proof (2) $\Rightarrow$ (1) • Let $\mathbb{B}_1, \mathbb{B}_2 \in \mathcal{C}$ be arbitrary such that



$$B_i = (B_1 \cap B_2) \cup \{b_i\} \quad (i \in \{1,2\})$$

and there is no amalgam in $\mathcal{C}$.

• τ binary $\Longrightarrow$ every potential amalgam $\mathbb{C}$ is obtained by specifying relations at $\{b_1, b_2\}$

no amalgam

$\Longrightarrow$ every choice $\mathbb{C}$ of relations at $\{b_1, b_2\}$ induces a substructure of $\mathbb{C}$ isomorphic to some $\mathbb{F} \in \mathcal{F}$.

i) There are at most ℓ such choices

ii) The witnessing substr. of $\mathbb{C}$ are of size $\leq m$

- i) + ii) $\mathbb{B}_1$ and $\mathbb{B}_2$ can be chosen of size $\leq (m-2) \cdot \ell$ $\blacksquare$

(!): This gives a silly $\mathrm{coNP}^{\mathrm{NP}}$ algorithm.

<u>Open problem</u>: Is the AP decidable for finitely bounded classes in general?

<u>2.3.6. Generic superpositions</u>

<u>Def</u> $\mathbb{A}_1, \mathbb{A}_2$ τ_1 and τ_2-structures, respectively, $\tau_1 \cap \tau_2 = \emptyset$.

- $\tau_1 \cup \tau_2$-structure $\mathbb{A}$ with $A = A_1 = A_2$ is the superposition of $\mathbb{A}_1$ and $\mathbb{A}_2$ if $\mathbb{A}|^{\tau_1} = \mathbb{A}_1$ and $\mathbb{A}|^{\tau_2} = \mathbb{A}_2$

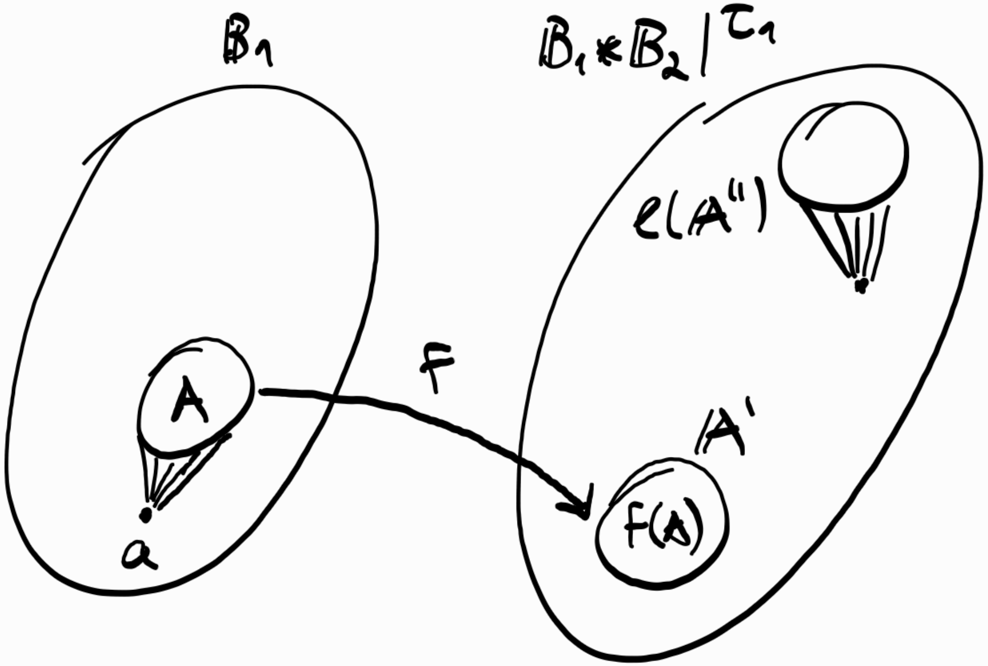
$\mathcal{C}_1$ and $\mathcal{C}_2$ classes of τ_1- and τ_2 structures, respectively

- the superposition of $\mathcal{C}_1$ and $\mathcal{C}_2$ is the class of all superpositions of $\mathbb{A}_1 \in \mathcal{C}_1$ and $\mathbb{A}_2 \in \mathcal{C}_2$

<u>Lemma 2.3.21</u> $\mathcal{C}_1, \mathcal{C}_2$ strong amalgamation classes with disjoint rel. signatures $\Rightarrow$ $\mathcal{C}_1 * \mathcal{C}_1$ is a strong amalgamation class

<u>Def</u> $\mathbb{B}_1, \mathbb{B}_2$ homogeneous str. with disjoint rel. signatures st. $\mathrm{Age}(\mathbb{B}_1)$ and $\mathrm{Age}(\mathbb{B}_2)$ have the SAP. Then the Fraïssé limit of $\mathrm{Age}(\mathbb{B}_1) * \mathrm{Age}(\mathbb{B}_2)$ is called the generic superposition of $\mathbb{B}_1$ and $\mathbb{B}_2$.

: $\mathbb{B}_1 * \mathbb{B}_2|^{\tau_1} \cong \mathbb{B}_1$, $\mathbb{B}_1 * \mathbb{B}_2|^{\tau_2} \cong \mathbb{B}_2$ (back & forth)



$\boxed{\text{Forth:}}$

1) $A' := \mathbb{B}_1 * \mathbb{B}_2[f(A)]$
2) $A'' :=$ any extension of A' by $\{a\}$ s.t. $\mathbb{B}_1[A \cup \{a\}] \cong A'|^{\tau_1}$

3) $\exists e : A'' \hookrightarrow \mathbb{B}_1 * \mathbb{B}_2$
4) homogeneity yields $f(a)$

$\boxed{\text{Back: trivial}}$

<u>Example 2.3.33</u> $\mathcal{C}_1, \mathcal{C}_2$ classes of $<_1$- and $<_2$-structures, respectively, where $<_i$ denotes a linear order. Then the Fraïssé limit of $\mathcal{C}_1 * \mathcal{C}_2$ is known as the random permutation.