RECAP of relevant MATERIAL (we work in first order logic)

$h: A \to B$ is

a HOMOMORPHISM if $\forall R \in \tau \quad A \models R(\bar{a}) \Rightarrow B \models R(h(\bar{a}))$.

an EMBEDDING if it is an injective homomorphism s.t.

$$\forall R \in \tau \quad A \models R(\bar{a}) \Leftrightarrow B \models R(h(\bar{a}))$$

an ELEMENTARY EMBEDDING if for all first order formula $\phi(\bar{a})$

$$A \models \phi(\bar{a}) \Leftrightarrow B \models \phi(h(\bar{a})).$$

$=$

PROP 2.1.14 tfae:

- $S_{\forall\text{-}} \subseteq T_{\forall\text{-}}$
- every model of T has a homomorphism to a model of S.

LEMMA 2.1.21

$\forall\exists^+$-formulas are preserved by direct limits of models of T.

HOMOMORPHISM PRESERVATION THEOREM

$\phi \equiv_T \exists^+$-formula iff it is preserved by all homomorphisms between models of T

ŁOŚ-TARSKI

$\phi \equiv_T \exists$-formula iff it is preserved by embeddings between models of T

§2.6.1 MODEL COMPLETENESS

T is MODEL COMPLETE if every embedding between models is ELEMENTARY

EXAMPLES:

- QE $\Rightarrow$ MODEL COMPLETENESS – embeddings guarantee preservation of qf-formulas; QE yiels that the embeddings are elementary.
- $Th(\mathbb{Z}, Succ)$ is model complete without QE, where $Succ = \{(x,y) \mid y = x+1\}$
- $Th(\mathbb{Q}^{\geq 0}; <)$ is not model complete — $\phi(x) := \forall y\,(x < y \vee x = y)$, $x \mapsto x+1$

EQUIVALENTS TO MODEL COMPLETENESS tfae:

1. T is model complete
2. every formula is $\equiv_T$ to an $\exists$-formula
3. for any embedding $h: A \to B$ between models of T, and ϕ an $\exists$-formula,
$$B \models \phi(h(\bar{a})) \Rightarrow A \models \phi(\bar{a})$$
4. every $\exists$-formula is $\equiv_T$ to a $\forall$-formula
5. every formula is $\equiv_T$ to a $\forall$-formula

MODEL COMPLETE THEORIES are $\forall\exists$-axiomatizable

§2.6.2 CORE THEORIES

B is a **CORE** if all endomorphisms of B are embeddings

T is a **CORE THEORY** if every homomorphism between models is an EMBEDDING

EXAMPLES:

- $Th(\mathbb{Q}, <)$ is a core theory $\quad a < b \iff h(a) < h(b)$
- $Th(\mathbb{Q}, \leq)$ is NOT a core theory – SEND EVERYTHING TO 0

EQUIVALENTS TO CORE THEORY tfae:

① T is a core theory

② $\exists$-formulas are $\equiv_T$ to $\exists^+$-formulas

③ For ψ atomic, $\neg\psi$ is equivalent to an $\exists^+$-formula

① ⇒ ② $\exists$ formulas are preserved by emb.

HOMS are EMB ⇒ $\exists$ fors are pres. by homs. ⇒ ②

② ⇒ ③ : $\neg\psi$ is $\exists$-form. ⇒ $\exists^+$-formula.

③ ⇒ ① : hom. preserve $\exists^+$-formulas. So homs are emb. ∎

The CONSTRAINT ENTAILMENT PROBLEM for T is the computational problem with

<u>INPUT</u>: Φ and Ψ in variables $x_1 \dots x_n$

Φ: pp-formula; Ψ: atomic formula

<u>QUESTION</u>: Does Φ ENTAIL Ψ? (i.e. do we have $T \models \forall x_1 \dots x_n (\Phi \to \Psi)$?)

equivalence to CSP(T) for CORE THEORIES

Let T be a core τ-theory for τ finite.

Then CEP(T) is equivalent to CSP(T) under poly-time Turing reduction

<u>Proof</u>: ($\Leftarrow$) CSP(T) asking whether given pp sentence ϕ $T \cup \{\phi\}$ is SAT.

CSP(T) is just the entailment problem

<u>INPUT</u> Φ & $\bot$ Q: $T \models \Phi \to \bot$?

($\Rightarrow$) T is a CORE $\Rightarrow$ $\neg\Psi$ is $\equiv_T$ to an $\exists^+$-formula

$\Psi_1 \vee \dots \vee \Psi_m$ of pp-formulas.

τ is finite m is bounded above by M across all possible Ψ.

$T \models \forall x (\Phi \to \Psi)$ iff $\boxed{T \cup \{\exists x (\Phi(x) \wedge \Psi_i(x))\} \text{ is NOT SATISFIABLE}}$ is a CSP

for all $i \leq m \leq M$

$\blacksquare$

§ 2.6.3 MODEL COMPLETE CORE THEORIES

what if T is both model complete & a core?

EQUIVALENTS TO BEING MODEL COMPLETE CORE THEORY tfae

1. T is a model complete core theory
2. formulas are $\equiv_T$ to $\exists^+$-formulas
3. For $h: A \to B$ a homomorphism between models of T, φ $\exists^+$-form
$$B \models \varphi(h(\bar{a})) \Rightarrow A \models \varphi(\bar{a})$$
4. $\exists^+$ formulas are $\equiv_T$ to $\forall^-$-formulas
5. formulas are $\equiv_T$ to $\forall^-$-formulas

T MODEL COMPLETE CORE THEORY $\Rightarrow$ T is equivalent to a $\forall\exists^+$-theory

§2.7 COMPANIONS

Perhaps T is NOT a model complete core, but we can find S s.t. $CSP(S) = CSP(T)$ which <u>is</u> a model complete core theory?

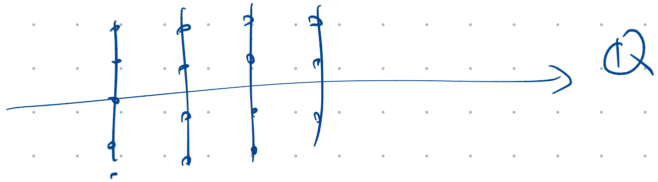
S is a **CORE COMPANION** of T if

- S is a model complete core theory
- $T_{\forall -} = S_{\forall -}$ (i.e. every model of S maps homomorphically to a model of T and vice-versa)

GUARANTEES $CSP(S) = CSP(T)$.

EXAMPLES:

- $Th(\mathbb{Q}, \leq)$ has CORE COMPANION $Th(\bullet, \leq)$. $CSP(\mathbb{Q}, \leq)$ is in P
- $Th(\mathbb{Z}, <)$ has CORE COMPANION $(\mathbb{Q}, <)$.



- Theory of undirected loopless graphs has CORE COMPANION $(\mathbb{N}, \neq)$.

Q: What does a CORE COMPANION look like (if it exists)?

- It is unique
- it corresponds to
  - POSITIVE KAISER HULL of T
  - theory of all T-epc structures for T.

=

a homomorphism $h: A \to B$ is an IMMERSION if for every $\exists^+$ formula φ, $B \models \varphi(h(\bar{a})) \Rightarrow A \models \varphi(\bar{a})$

A is EXISTENTIALLY POSITIVE CLOSED for T (T-epc) if there is a homomorphism from A to a model of T and every homomorphism $h: A \to B \models T$ is an immersion.

T-epc iff $T_{\forall^-}$-epc: A structure is T-epc iff it is $T_{\forall^-}$-epc.

WE CAN "CONTINUE" MODELS TO T-epc structures

Let $\kappa \geq \max(|L|, \aleph_0)$. Every model of T of cardinality $\leq \kappa$ admits a homomorphism to a T-epc structure of card. $\leq \kappa$

§2.7.2 positive KAISER HULL

EXISTENCE Let T be a first-order theory. There is a UNIQUE LARGEST $\forall\exists^+$-theory T' s.t. $T'_{\forall^-} = T_{\forall^-}$

we call T' the **POSITIVE KAISER HULL** of T, T^{KH^+}

proof:

Suppose by CONTRADICTION there are S and S' $\forall\exists^+$-theories s.t.

ⓐ $S_{\forall^-} = S'_{\forall^-} = T_{\forall^-}$

ⓑ $S \cup S'$ is unsatisfiable.

ⓐ $\Rightarrow$ we can build a coherent homomorphisms $f_{ij} : A_i \longrightarrow A_j$

s.t. $\begin{cases} \text{for } i \text{ even} & A_i \models S \\ \text{for } i \text{ odd} & A_i \models S' \end{cases}$

$B := \lim\limits_{i<\omega} A_i \models S \cup S'$ since $\forall\exists^+$-formulas are preserved by direct limits $\times\!\!\times$

$\blacksquare$

T^{KH^+} = COMMON $\forall\exists^+$ theory of T-epc structures

The positive KAISER HULL of T is the set of $\forall\exists^+$-sentences holding in every T-epc structure.

§2.7.3 CORE COMPANIONS

EQUIVALENTS TO HAVING A CORE COMPANION tfae

1. T has a core companion
2. All models of T^{KH^+} are T-epc
3. The class of T-epc structures is first-order axiomatizable

In particular, if T has a CORE COMPANION T^*,

- $T^* \equiv$ theory of all T-epc structures
- $T^* \equiv T^{KH^+}$

Proof: ① $\Rightarrow$ ②: Let U be the core companion of T.

U is $\equiv$ to a $\forall\exists^+$-theory and so, by $U_{\forall^-} = T_{\forall^-}$, $U \subseteq T^{KH^+}$ by def of T^{KH^+}

So, it is sufficient to show: $A \models U \Rightarrow A$ is T-epc.

- A maps hom to a model of T by $U_{\forall^-} = T_{\forall^-}$
- Let $h : A \to B \models T$ be a homomorphism. Say $B \models \varphi^{\exists^+}(h(\bar{a}))$.

$B \xrightarrow{g} C \models U$ by $U_{\forall^-} = T_{\forall^-}$. So $C \models \varphi(gh(\bar{a}))$.

U is a MODEL COMPLETE CORE $\Rightarrow g \circ h$ is an elementary embedding $\Rightarrow A \models \varphi(\bar{a})$ so A is T-epc.

2 $\Rightarrow$ 3 :

we know A is T-epc $\Rightarrow A \models T^{KH+}$.

so, if $\underbrace{A \models T^{KH+} \Rightarrow A \text{ is } T\text{-epc}}_{2}$, T^{KH+} axiomatizes T-epc structures

3 $\Rightarrow$ 1 : Suppose class of T-epc structures is axiomatized by U

• $U_{\forall^-} = T_{\forall^-}$:

• Any model of U has a homom to a model of $T \Rightarrow T_{\forall^-} \subseteq U_{\forall^-}$

• Any model of T can be contined to a $\underbrace{T\text{-epc model}}_{\text{model of } U} \Rightarrow U_{\forall^-} \subseteq T_{\forall^-}$.

• U is a model complete core theory

this is equivalent to: every homomorphism between models of U is an immersion.

Let $h: A^{\models U} \to B^{\models U}$ be a homomorphism and ϕ be a $\exists^+$-formula

$B \models \phi(h(\bar{a}))$. $T_{\forall^-}$-epc theory $B \models T_{\forall^-}$

$B \models T_{\forall^-}$ + A is $T_{\forall^-}$-epc $\Rightarrow h$ is an immersion

by def of $T_{\forall^-}$-epc ∎

An extra for model theorists:

PRESERVATION of model theoretic properties

model theorists are interested in model theoretic properties such as STABLE, NIP, simple etc.

Q: Let S be the core companion of T,

1. does S preserve model theoretic properties of T?
2. does T have the model theoretic properties of S?

A1: overall, yes!

Let $XP \in \{OP, IP, K\text{-}TP, K\text{-}TP_2, SOP_1, SOP_2\}$.

Then, T has $NXP \Rightarrow S$ has NXP

think of $(\mathbb{Q}, \leq)$ and point

A2: NO, but we do have

$T_{\forall^-} = S_{\forall^-} \Rightarrow T$ has NXP^+ iff S has NXP^+

where XP^+ is the positive version of XP.

See, for example

DIVIDING LINES BETWEEN POSITIVE THEORIES, by DIMITRIEVA, GALLINARO, KAMSMA