CSP Reading Group

Sections 3.1–3.2

3.1 INTRODUCING PP-INTERPRETATIONS

DEF 2.4.1: $\mathbb{A}, \mathbb{B}$ rel. structures d ... dimension

I partial surjection $A^d \to B$ is a pp-interpretation of $\mathbb{B}$ in $\mathbb{A}$ if for every R of arity k defined by an atomic fla ϕ in $\mathbb{B}$

$I^{-1}(R) \subseteq A^{dk}$ has a pp-definition ϕ_I in $\mathbb{A}$

OBSERVATION:

T_I ... domain of I $=_I$ ... kernel of I

full interpretation:

$R \subseteq B^k$ is pp-definable in $\mathbb{B}$ iff $I^{-1}(R)$ is in $\mathbb{A}$

EXAMPLE 3.1.3:

$(\mathbb{Q}; 0, 1, +, *)$ has a 2-dim. pp-interpretation in $(\mathbb{Z}; 0, 1, +, *)$

(+ and * are graphs of the operations)

Fact: $\mathbb{Z}_{\geq 0}$ is expressed by

$$\phi(x) = \exists x_1, x_2, x_3, x_4 \; \underbrace{(x = x_1^2 + x_2^2 + x_3^2 + x_4^2)}$$

doable with an existential fla

$T_I(x, y)$: $\quad y \geq 1 \iff \exists z \;\; z + 1 = 0 \wedge \phi(y + z)$

$=_I(x_1, y_1, x_2, y_2)$: $\quad x_1 y_2 = y_1 x_2$

etc.

Why pp-interpretations? Reductions!

THEOREM 3.1.4

$\mathbb{A}, \mathbb{B}$ rel. structures with finite signatures

$\exists$ pp-interpretation of $\mathbb{A}$ in $\mathbb{B}$ $\Rightarrow$

$\Rightarrow \exists$ poly-time reduction from $CSP(\mathbb{A})$ to $CSP(\mathbb{B})$

Proof idea: Translate the constraints and ensure by constraints $T_I(\dots)$ that the solution is in the domain of I. ■

REMARK: We may <u>compose pp-interpretations</u> in a natural way.

COROLLARY 3.1.6:

If $\mathbb{B}$ pp-interprets [a hard structure e.g. K_3, $(\{0,1\}, 1IN3)$, $(\{0,1\}, NAE)$] then $\mathbb{B}$ has a finite-signature reduct with an NP-hard CSP.

Proof: The pp-interpretation uses only finitely many relations. + Theorem 3.1.4. ∎

⇝ key to "natural" hardness proofs that are stronger than just pp-definitions

PROPOSITION 3.1.7:

$\mathbb{A}$ rel. structure with finite relational signature

$\mathbb{B}$, $c_1, \dots, c_k \in B$ such that

- the orbit of $(c_1, \dots, c_k)$ under $\mathrm{Aut}(\mathbb{B})$ is pp-definable
- $\mathbb{A}$ has a pp-interpretation in $(\mathbb{B}, c_1, \dots, c_k)$

$\Rightarrow \exists$ finite-signature reduct $\mathbb{B}'$ of $\mathbb{B}$ and a poly-time reduction from $\mathrm{CSP}(\mathbb{A})$ to $\mathrm{CSP}(\mathbb{B}')$

In particular: $\mathbb{B}$ has finite signature $\Rightarrow$ we can take $\mathbb{B}' = \mathbb{B}$

$\rightsquigarrow$ We can use this techique to prove hardness!

$T_3 := \{(x,y,z) \in \mathbb{Q}^3 \mid (x=y<z) \vee (x=z<y)\}$

PROPOSITION 3.1.9:

$(\mathbb{Q}; T_3, 0)$ pp-interprets $(\{0,1\}, 1IN3)$. $\rightarrow \{(0,0,1), (0,1,0), (1,0,0)\}$

Prop 3.1.7 $\Rightarrow$ $CSP(\mathbb{Q}, T_3)$ is NP-hard.

$\rightsquigarrow$ the orbit of 0 is $\mathbb{Q}$

Proof: 2-dim. interpretation

domain fla T_I is $T_3(0, x_1, x_2)$

$T_I(b_1, b_2)$ => one of the elements is 0 and the other is >0

$$I(b_1, b_2) = \begin{cases} 1 & \text{if } b_1 = 0 \\ 0 & \text{otw.} \end{cases}$$

$=_I(x_1, x_2, y_1, y_2)$ is $T_3(0, x_1, y_2)$ $\rightsquigarrow$ the second coordinate is needed here

We claim

$1IN3_I(x_1, x_2, y_1, y_2, z_1, z_2)$ is

$\exists u, v, w \ (T_3(0, u, z_1) \wedge T_3(u, v, y_1) \wedge T_3(v, w, x_1) \wedge T_3(0, 0, w))$

Let $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{Q}$.

$t := (I(x_1, x_2), I(y_1, y_2), I(z_1, z_2))$

Suppose that $t \in 1IN3$:

1) $t = (1, 0, 0) \Rightarrow x_1 = 0, \ y_1, z_1 > 0 \Rightarrow u, v := 0, \ w := 1$ ✓

2) $t = (0, 1, 0) \Rightarrow x_1, z_1 > 0, \ y_1 = 0 \Rightarrow u := 0, \ w := \frac{x_1}{2}, \ v := w$ ✓

3) $t = (0, 0, 1) \Rightarrow x_1, y_1 > 0, \ z_1 = 0 \Rightarrow w := \min\{\frac{x_1}{2}, \frac{y_1}{2}\}$, $v := w, \ u := v$ ✓

We claim

$1IN3_F(x_1, x_2, y_1, y_2, z_1, z_2)$ is

$$\exists u, v, w \; \big(T_3(0, u, z_1) \wedge T_3(u, v, y_1) \wedge T_3(v, w, x_1) \wedge T_3(0, 0, w)\big)$$

Suppose $(x_1, x_2, y_1, y_2, z_1, z_2)$ satisfies the fla:

$$T_3(0,0,w) \Rightarrow w > 0$$

$$T_3(0,u,z_1) \begin{cases} u = 0 \;\&\; z_1 > 0 \xRightarrow{T_3(u,v,y_1)} \begin{cases} v = 0,\ y_1 > 0 \xRightarrow{T_3(v,w,x_1)} x_1 = 0 \Rightarrow t = (1,0,0) \\ v > 0,\ y_1 = 0 \xRightarrow{T_3(v,w,x_1)} x_1 > 0 \Rightarrow t = (0,1,0) \end{cases} \\ u > 0 \;\&\; z_1 = 0 \xRightarrow{T_3(u,v,y_1)} v > 0 \;\&\; y_1 > 0 \xRightarrow{T_3(v,w,x_1)} x_1 > 0 \Rightarrow t = (0,0,1) \end{cases}$$

$\blacksquare$

$\mathrm{Betw} := \{(x,y,z) \in \mathbb{Q}^3 \mid x<y<z \vee z<y<x\}$

PROPOSITION 3.1.10:

$(\mathbb{Q}; \mathrm{Betw}, 0)$ pp-interprets $(\{0,1\}, \mathrm{NAE})$ $\rightarrow \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$

PROP 3.1.7 $\Longrightarrow$ $\mathrm{CSP}(\mathbb{Q}, \mathrm{Betw})$ is NP-hard (the orbit 0 is $\mathbb{Q}$).

Proof: dimension $= 1$

domain formula T_I is $\exists z\, \mathrm{Betw}(x,0,z)$ i.e. $x \neq 0$

$$I(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$$ $\rightsquigarrow$ equality corresponds to having the same sign

$=_I(x,y)$ is $\exists z\,(\mathrm{Betw}(x,0,z) \wedge \mathrm{Betw}(y,0,z))$ i.e.

$(x>0 \Leftrightarrow y>0)$

$$\mathrm{NAE}_I(x,y,z) \text{ is } \exists u\, (\underbrace{\mathrm{Betw}(x,u,y)}_{\text{they can't both have the same sign as } z} \wedge \underbrace{\mathrm{Betw}(u,0,z)}_{u \text{ has a different sign than } z})$$

$\blacksquare$

3.2. PP-INTERPRETING ALL FINITE STRUCTURES

COROLLARY 3.2.1

K_3 pp-interprets all finite structures.

Proof:

Section 6 => every fo-fla is equivalent to a pp-fla

Lemma 2.4.4 => every finite structure has an fo-interpretation in K_3

■

⇝ towards a universal reason for hardness for finite-domain CSPs ... "expressing K_3, $(\{0,1\}, 1IN3)$, ..."

THEOREM 3.2.2

TFAE for $\mathbb{B}$:

(1) $\mathbb{B}$ pp-interprets $(\{0,1\}, 1IN3)$

(2) — || — $(\{0,1\}, NAE)$

(3) — || — K_n for some $n \geq 3$

(4) — || — a structure $\mathbb{A}$, $|A| \geq 2$ and all fo-formulas are equivalent to pp-formulas over $\mathbb{A}$

(5) — || — all finite structures

Proof: (5) => (1), (2), (3)

In Section 6 we get (1) -> (2), (2) -> (4) and (3) => (4).

(4) => (5) is by Lemma 2.4.4 (a structure with ≥ 2 elements fo-interprets all finite structures). ■

Is this "the only reason for hardness" for $CSP(\mathbb{A})$ for finite $\mathbb{A}$?

$\underbrace{\mathbb{A} \rightleftarrows K_3}_{}$ $\Rightarrow$ $CSP(\mathbb{A}) = CSP(K_3) \Rightarrow CSP(\mathbb{A})$ is hard

maps homomorphically in both directions = homomorphically equivalent

$\leadsto$ we need to add homomorphic equivalence on the top of pp-interpretations

$\leadsto$ pp-constructions (section 3.6)